

**Tangent-point repulsive potentials
for a class of non-smooth m -dimensional sets in \mathbb{R}^n .
Part I: Smoothing and self-avoidance effects**

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Abstract

We consider repulsive potential energies $\mathcal{E}_q(\Sigma)$, whose integrand measures tangent-point interactions, on a large class of non-smooth m -dimensional sets Σ in \mathbb{R}^n . Finiteness of the energy $\mathcal{E}_q(\Sigma)$ has three sorts of effects for the set Σ : topological effects excluding all kinds of (a priori admissible) self-intersections, geometric and measure-theoretic effects, providing large projections of Σ onto suitable m -planes and therefore large m -dimensional Hausdorff measure of Σ within small balls up to a uniformly controlled scale, and finally, regularizing effects culminating in a geometric variant of the Morrey-Sobolev embedding theorem: Any admissible set Σ with finite \mathcal{E}_q -energy, for any exponent $q > 2m$, is, in fact, a C^1 -manifold whose tangent planes vary in a Hölder continuous manner with the optimal Hölder exponent $\mu = 1 - (2m)/q$. Moreover, the patch size of the local $C^{1,\mu}$ -graph representations is uniformly controlled from below only in terms of the energy value $\mathcal{E}_q(\Sigma)$.

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1 Introduction

This paper grew out of a larger project, devoted to the investigation of so-called *geometric curvature energies* which include various types of geometric integrals, measuring the degree of smoothness and bending for objects that do not, at least a priori, have to be smooth. Here, we study the energy functional

$$\mathcal{E}_q(\Sigma) = \int_{\Sigma} \int_{\Sigma} \frac{1}{R_{\text{tp}}^q(x, y)} d\mathcal{H}^m(x) d\mathcal{H}^m(y) \quad (1.1)$$

defined for a class \mathcal{A} of admissible, m -dimensional sets in \mathbb{R}^n . The precise definition of \mathcal{A} is given in Section 2; we just mention now that for each $\Sigma \in \mathcal{A}$ a weak counterpart of the classic tangent plane is defined almost everywhere with respect to the m -dimensional Hausdorff measure \mathcal{H}^m on Σ . In other words, for \mathcal{H}^m -a.e. $x \in \Sigma$ there is an m -plane H_x such that the portion of Σ near the point x is close to the affine plane $x + H_x \subset \mathbb{R}^n$. The quantity

$$R_{\text{tp}}(x, y) := \frac{|y - x|^2}{2 \text{dist}(y, x + H_x)} \quad (1.2)$$

in the integrand is referred to as the *tangent-point radius* and denotes the radius of the smallest sphere tangent to the affine plane $x + H_x$ and passing through y . (If y happens to be contained in $x + H_x$, then we set $1/R_{\text{tp}}(x, y) = 0$.) Thus, $1/R_{\text{tp}}(x, y)$ is defined a.e. on $\Sigma \times \Sigma$ with respect to the product measure $\mathcal{H}^m \otimes \mathcal{H}^m$. Notice that for any compact embedded manifold of class $C^{1,1}$ this repulsive potential \mathcal{E}_q is finite. For two-dimensional surfaces in \mathbb{R}^3 , i.e. $n = 3, m = 2$, Banavar et al. [2] suggested, in fact, the use of such tangent-point functions to construct self-interaction energies with non-singular integrands that do not require any sort of ad hoc regularization, in contrast to standard repulsive potentials. The latter would penalize any two surface points that are close in Euclidean distance, no matter whether these points are adjacent on the surface (leading to singularities) or belong to different sheets of the same surface. Our aim here is to show that for the infinite range of exponents $q > 2m$ finiteness of $\mathcal{E}_q(\Sigma)$ has three sorts of consequences for any admissible set $\Sigma \in \mathcal{A}$: measure-theoretic, topological, and analytical. To see them in a proper perspective, let us give a plain description of the surfaces we work with.

Our class \mathcal{A} consists of m -dimensional sets $\Sigma \in \mathbb{R}^n$ with finite measure $\mathcal{H}^m(\Sigma) < \infty$ on which we impose (1) a certain degree of flatness in the neighbourhood of many (but a priori not all!) points of Σ , and (2) some degree of connectivity. A priori, we allow for various self-intersections of Σ , and for singularities along low dimensional subsets. For the purposes of this introduction, however, it is enough to keep in mind the following examples of admissible surfaces (more general examples are presented in Section 2.3):

- (i) If $\Sigma_0 = M_1 \cup \dots \cup M_N$, where $N \in \mathbb{N}$ is arbitrary and all $M_i \subset \mathbb{R}^n$ are compact, closed, embedded m -dimensional submanifolds of class C^1 such that $\mathcal{H}^m(M_i \cap M_j) = 0$ whenever $i \neq j$, then Σ_0 is admissible;
- (ii) If Σ_0 is as above, then $\Sigma_1 = F(\Sigma_0)$ is admissible whenever F is a bilipschitz homeomorphism of \mathbb{R}^n .

The dimension m and the codimension $n - m$ of Σ in \mathbb{R}^n are fixed throughout the paper but otherwise arbitrary. The reader may adopt for now the temporary definition

$$\mathcal{A} := \{\Sigma \subset \mathbb{R}^n : \Sigma = F(\Sigma_0), \Sigma_0 \text{ as in (i) above, } F: \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ bilipschitz}\}.$$

It is easy to see that $q_0 = 2m = \dim(\Sigma \times \Sigma)$ is a critical exponent here: for $q = q_0$ the energy $\mathcal{E}_q(\Sigma)$ is scale invariant, and for each $q \geq q_0$ a surface Σ with a conical singularity at one point must have $\mathcal{E}_q = \infty$. We prove in this paper that for $q > q_0 = 2m$ all kinds of singularities are excluded. In fact, upper bounds for $\mathcal{E}_q(\Sigma)$ lead to three kinds of effects. Firstly, measure-theoretic effects: the measure of Σ contained in a ball of radius r is comparable to r^m on small scales that depend solely on the energy. Secondly topological effects: an admissible surface Σ with finite \mathcal{E}_q -energy has no self-intersections, it must be an embedded manifold, and finally, far-reaching analytical consequences: we have precise $C^{1,\mu}$ bounds for the charts in an atlas of Σ .

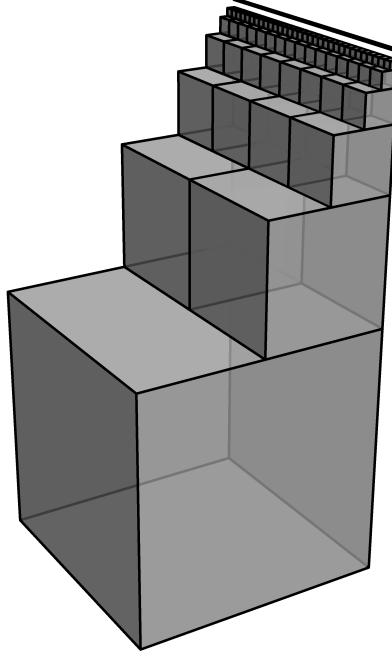


Figure 1: An infinite stack of cubes, also an admissible surface (which turns out to have infinite \mathcal{E}_q -energy for all $q > 2m = 4$).

Let us first state the results precisely and then comment on the proofs and discuss the relations of this paper to existing research.

Remark 1.1. *Keep in mind, though, that all results stated in the introduction will be proved for a more general class $\mathcal{A}(\delta)$ of admissible sets much larger than the preliminary class \mathcal{A} defined above; see Section 2.3. To get a first impression of other admissible sets have a look at Figure 1.*

Theorem 1.2 (Uniform Ahlfors regularity). *Assume that $\Sigma \in \mathcal{A}$ is an admissible m -dimensional surface in \mathbb{R}^n with $\mathcal{E}_q(\Sigma) \leq E$, $q > 2m$. There exists a constant $a_1 = a_1(q, n, m) > 0$, depending only on q, n and m , such that*

$$\mathcal{H}^m(\Sigma \cap B(x, r)) \geq \frac{1}{2} \omega(m) r^m$$

for all $x \in \Sigma$ and all radii

$$0 < r < R_1 \equiv R_1(q, n, m, E) := \frac{a_1}{E^{1/(q-2m)}}.$$

(Here, and throughout the paper, $B(x, d)$ denotes the closed ball of radius d centered at x .)

In other words: if $\Sigma \in \mathcal{A}$ has finite energy for some $q > 2m$, then up to the length scale given by R_1 – which depends *only* on the energy bound E and the parameters m, n, q , but *not* on Σ itself – isolated *thin fingers*, narrow tubes, and the like cannot form on Σ . The measure of the portion of Σ inside the ball $B(x, r)$ is at least as large as half of the measure of the m -dimensional equatorial cross-section of $B(x, r)$. A similar lower estimate on the Ahlfors regularity was proven by L. Simon

for smooth two-dimensional surfaces with finite Willmore energy [30, Corollary 1.3]; see also the work of P. Topping [38] which even contains *sharp* lower bounds for the sum of local L^2 -norm of the classic mean curvature and the area of the surface in a small ball. Mean curvature at a particular point x on a smooth surface in \mathbb{R}^3 may be viewed as the arithmetic mean of minimal and maximal normal curvature at x . R_{tp}^{-1} , on the other hand, is a two-point function taking non-local interactions into account as well, but if one looks at the coalescent limits $\lim_{y \rightarrow x} R_{\text{tp}}^{-1}(x, y)$ one obtains absolute values of intermediate normal curvatures at x depending on the direction of approach as y tends to x (cf. [2, Section 3.2]). So, the local portion of our energy \mathcal{E}_q near x may be regarded as another kind of averaging normal curvatures at x , leading to density estimates as does the Willmore functional.

The next result gives a quantitative description of flatness of Σ , in terms of the so-called β -numbers introduced by P. Jones.

Theorem 1.3 (Uniform decay of β -numbers). *Let $\Sigma \in \mathcal{A}$ be an admissible m -dimensional surface in \mathbb{R}^n with $\mathcal{E}_q(\Sigma) < E$ for some $q > 2m$. There exist two constants $a_2(q, n, m) > 0$ and $A_2(q, n, m) < \infty$, both depending only on n, m and q , such that whenever the radius*

$$d \leq R_2 \equiv R_2(q, n, m, E) := \frac{a_2(q, n, m)}{E^{1/(q-2m)}} \quad (1.3)$$

and the bound $\varepsilon > 0$ satisfy the balance condition

$$\varepsilon^{4m+q} d^{2m-q} \geq A_2(q, n, m) E, \quad (1.4)$$

then we have

$$\beta_\Sigma(x, d) := \inf_{P \in G(n, m)} \left(\sup_{y \in B(x, d) \cap \Sigma} \frac{\text{dist}(y, x + P)}{d} \right) \leq \varepsilon, \quad x \in \Sigma, \quad (1.5)$$

where $G(n, m)$ denotes the Grassmannian of all m -dimensional subspaces of \mathbb{R}^n .

Thus, for small d we have

$$\beta_\Sigma(x, d) \lesssim E^{1/(4m+q)} d^\kappa, \quad \kappa := \frac{q-2m}{q+4m} > 0.$$

It is known that this condition alone does not suffice to conclude that Σ is a topological manifold. D. Preiss, X. Tolsa and T. Toro [24], extending an earlier work of G. David, C. Kenig and T. Toro [7], study Reifenberg flat sets Σ whose β -numbers satisfy such estimates, see e.g. [24, Prop. 2.4] where it is proved that a decay bound for β 's combined with Reifenberg flatness¹ implies that Σ must be a submanifold of class $C^{1, \kappa}$.

Since $\Sigma \in \mathcal{A}$ might, at least a priori, have transversal self-intersections, we do not have Reifenberg flatness here, and a quick direct use of the results of [7, 24] is impossible. However, we are able to use the energy estimates and the information given by Theorem 1.3 iteratively. Extending the ideas from our earlier work [34, Section 5] devoted to surfaces in \mathbb{R}^3 , we prove here that at every point $x \in \Sigma$ there exists the classic tangent plane $T_x \Sigma$, and that the oscillation of tangent planes along Σ satisfies uniform Hölder estimates. This implies that each $\Sigma \in \mathcal{A}$ with $\mathcal{E}_q(\Sigma) < \infty$ must be an embedded m -dimensional manifold of class $C^{1, \kappa}$. Later on, working with graph patches of Σ , we use slicing techniques and a bootstrap reasoning to improve and sharpen this information. The following theorem is the main result of this paper.

¹We do not define that condition here since we will not work with it directly; let us just mention that Reifenberg flatness means that the rescaled Hausdorff distance between $\Sigma \cap B(x, d)$ and an m -plane P in $B(x, d)$ is uniformly controlled, and small.

Theorem 1.4 (Geometric Sobolev–Morrey imbedding). *Let $\Sigma \in \mathcal{A}$ and $\mathcal{E}_q(\Sigma) < +\infty$ for some $q > 2m$. Then Σ is an embedded submanifold of class $C^{1,\mu}$, where $\mu = 1 - 2m/q$.*

In fact, there exist constants $a_3, A_3 > 0$, depending only on m, n , and q , with the following property: For each $x \in \Sigma$ and each $r \leq R_3 = a_3 \mathcal{E}_q(\Sigma)^{1/(q-2m)}$ there exists an m -plane $P \in G(n, m)$ and a function $f: P \simeq \mathbb{R}^m \rightarrow P^\perp \simeq \mathbb{R}^{n-m}$ of class $C^{1,\mu}$ such that

$$\Sigma \cap B(x, r) = \Sigma \cap \text{graph } f,$$

where $\text{graph } f \subset P \times P^\perp \simeq \mathbb{R}^n$ denotes the graph of f , and

$$|\nabla f(z) - \nabla f(w)| \leq A_3 E(x, r)^{1/q} |z - w|^\mu, \quad z, w \in P \cap B(0, r), \quad (1.6)$$

where

$$E(x, r) := \int_{B(x, r) \cap \Sigma} \int_{B(x, r) \cap \Sigma} \left(\frac{1}{R_{\text{tp}}(u, v)} \right)^q d\mathcal{H}^m(u) d\mathcal{H}^m(v).$$

We believe that the exponent $\mu = 1 - 2m/q$, strictly larger than $\kappa = (q - 2m)/(q + 4m)$, is optimal here. It is clear that finiteness of \mathcal{E}_q does not lead to C^2 regularity: consider a rotational cylinder closed with two hemispherical caps as an admissible surface Σ of class $C^{1,1}$ but not in C^2 . For this particular surface inequality (1.6) is qualitatively optimal and, due to the factor $E(x, r)^{1/q}$ and boundedness of $1/R_{\text{tp}}$, yields in fact *Lipschitz* estimates for the gradient of local graph representations of Σ .

Please note two more things. First, the exponent $\mu = 1 - 2m/q$ is computed according to the recipe used in the classic Sobolev–Morrey imbedding theorem in the supercritical case. Here, the dimension of the domain of integration, i.e. of $\Sigma \times \Sigma$, equals $2m$. We have $\mu \rightarrow 1$ as $q \rightarrow \infty$; for two-dimensional surfaces, the limiting case $q = \infty$ has been treated earlier in our papers [32] and [33].

Second, what we have learnt about Σ is not limited to embeddedness and purely qualitative $C^{1,\mu}$ estimates. It is clear that the bounds given by Theorem 1.4 are uniform in any class of surfaces with uniformly bounded energy \mathcal{E}_q . In other words, if $\mathcal{H} = \{\Sigma_i : i \in I\} \subset \mathcal{A}$ satisfies

$$\sup_{i \in I} \mathcal{E}_q(\Sigma_i) \leq M < \infty, \quad (1.7)$$

then we can find two constants $A, \delta > 0$, depending only on M, m, n and q , such that each $\Sigma_i \cap B(x, \delta)$, where $i \in I$ and $x \in \Sigma_i$, is obtained by a rigid motion of \mathbb{R}^n from a graph of a function $f: \mathbb{R}^m \rightarrow (\mathbb{R}^m)^\perp \simeq \mathbb{R}^{n-m}$ which satisfies the uniform estimate $\|f\|_{C^{1,\mu}} \leq A$, no matter how $i \in I$ and $x \in \Sigma_i$ have been chosen. Thus, a uniform upper bound on \mathcal{E}_q allows us to fix a uniform size of charts for all $\Sigma_i \in \mathcal{H}$, and forces the equicontinuity of gradients of local graph representations of the surfaces Σ_i . In a forthcoming paper [36] we show how to use this idea to obtain finiteness theorems for classes of C^1 embedded manifolds Σ_i in \mathbb{R}^n satisfying a volume constraint and a uniform energy bound (1.7).

We do not know what happens in the critical case $q = 2m$. Let us mention here one plausible conjecture that we cannot prove at this stage.

Conjecture 1.5. *Every immersed m -dimensional C^1 -manifold in $\Sigma \subset \mathbb{R}^n$ with finite \mathcal{E}_{2m} -energy is embedded.*

Another, probably more difficult, question that we cannot handle at present is the following: how regular are the minimizers of \mathcal{E}_q (say, with upper bounds for the total measure, to prevent the decrease of energy caused by rescaling) in isotopy classes of C^1 embedded manifolds? Are they $C^{1,1}$ (this is optimal for *ideal links* [5] – corresponding to the case $q = \infty$ in dimensions $n = 3$ and $m = 1$ – where

contact phenomena are present)? Or maybe C^∞ , as minimizers of a Möbius invariant knot energy in [9], [14]; see also [25], [26]? In addition, S. Blatt [3] characterized all curves with finite Möbius energy as embeddings in certain Sobolev-Slobodeckii classes; such a *characterization* of finite energy submanifolds for the tangent-point energy \mathcal{E}_q is presently not known.

Our interest in this topic has been triggered by several factors. They include manifold applications of Menger curvature in harmonic analysis and geometric measure theory (see e.g. the survey articles of P. Mattila [21], [22], G. David [6] and X. Tolsa [37], and the literature cited therein, including J.C. Léger [18] and the relation between 1-rectifiability and L^2 -integrability of Menger curvature). There are also works of different origin, investigating another geometric concept, the so-called global curvature introduced by Gonzalez and Maddocks [12]. The second author of the present paper took part in laying out the strict mathematical foundations for global curvature of rectifiable loops and its variational applications to elastic curves and rods with positive thickness; see [13], [27], [28], [29], [10], [11]. Part of this work, in turn, has been a starting point for our subsequent joint research devoted to various energies that, roughly speaking, interpolate between global curvature and Menger curvature. Finiteness of these energies; see e.g. [31], [34], [35], analogously to the case that we consider here, leads to an increase of regularity, to compactness effects, and yields a tool to control the amount of bending of non-smooth objects in purely geometric terms.² The novelty in the present paper is that we work in full generality, overcoming the difficulty that both the dimension and the codimension may be arbitrary. In an ongoing research [16, 17] S. Kolasiński obtains analogues of our results for basically the same admissible class of surfaces that we consider here, but for a different integral energy, defined as an $(m+2)$ -fold integral (with respect to \mathcal{H}^m) over the set of all simplices with vertices on Σ , directly extending our results in [34] to surfaces of arbitrary dimension and codimension.

Closely related research includes also G. Lerman and J.T. Whitehouse [19], [20], who investigate a number of ingenious high-dimensional curvatures of Menger type and obtain rectifiability criteria for d -dimensional subsets of Hilbert spaces. Last but not least, the deep and classic paper of W. Allard [1] sets forth a regularity theory for m -dimensional varifolds whose first variation (roughly: the distributional counterpart of mean curvature) is in L^p for some $p > m$. Our regularity results bear some resemblance to his Theorem 8.1. There are many differences, though, that remain to be fully understood. It is clear that without some extra topological assumptions on Σ finiteness of (1.1) cannot lead to the conclusion that Σ is locally (on a scale depending only on the energy!) homeomorphic to a disc; one could punch an arbitrary number of ‘holes’ in a smooth surface and this would just decrease the energy we work with. In Allard’s case, once we fix a ball where appropriate density estimates hold and the weight $\|V\|$ of the varifold V is close to the Hausdorff measure of a disk, then the ‘lack of holes’ is built into his assumption on the first variation δV of V . On the other hand, \mathcal{E}_q – as a non-local energy in contrast to the locally defined distributional mean curvature – averages over all global tangent-point interactions, which leads to self-avoidance and control over topology of the given surface. Admissible sets with finite \mathcal{E}_q -energy are differentiable manifolds, which Allard’s result cannot guarantee for varifolds with distributional mean curvature in L^p , $p > m$: there is a remaining (small) singular set, such that there is no control on the topology of the support of the varifold measure. To possibly bridge the apparent gap between Allard’s work and our results we should note that versions of \mathcal{E}_q can be defined for general m -dimensional varifolds V , via double integrals: the integrand $1/R_{\text{tp}}$ can be treated as a function on *points and planes*. It is an intriguing question whether finiteness of such integrals for some q ’s lead to rectifiability criteria or to an improved regularity in the case of varifolds.

²In [35] we treat the toy case $m = 1$ of the present paper, along with a few knot-theoretic applications of \mathcal{E}_q for curves.

Let us now informally sketch the main thread of our reasoning, and describe the organization of the paper in more detail. We want to exclude self intersections and to have a quantitative description of flatness; for this, Theorem 1.3 would be a good starting point. The main idea behind its proof is pretty straightforward: if the β -numbers were too large, i.e. if $x \in \Sigma$ but $\Sigma \cap B(x, d)$ were not confined to a narrow tube $B_{\varepsilon d}(x + P)$ around some affine m -plane $x + P$, then, a simple argument shows that we would have two much smaller balls $B_1, B_2 \subset B$, say with

$$\text{diam} B_1 = \text{diam} B_2 \approx \varepsilon^2 d$$

such that for all $y \in \Sigma \cap B_2$ and a nonzero proportion of $z \in \Sigma \cap B_1$ the distance $\text{dist}(y, z + T_z \Sigma)$ would be comparable to εd . This yields $1/R_{\text{tp}}(y, z) \gtrsim \varepsilon/d$, and a lower bound for the energy follows easily, leading to a contradiction, if the B_1, B_2 and the bound for the β 's are chosen in a suitable way which happens to be precisely the balance condition (1.4). There is only one serious catch here: in order to make the resulting estimate uniform, and to be able to iterate it later on, we must guarantee that

$$\mathcal{H}^m(\Sigma \cap B_r) \geq c \cdot r^m \quad \text{for all } r < r_0 = r_0(\text{energy}),$$

with some *absolute* constant c . And we want *both* r_0 and c independent of a particular Σ .

For this, we need Theorem 1.2 which serves as the backbone for all the later constructions and estimates of the paper. The overall idea here is somewhat similar to an analogous result in our work [34] on Menger curvature for surfaces in \mathbb{R}^3 . The main difference, however, leading to crucial difficulties, is that the *codimension* of Σ may be arbitrary.

The proof of Theorem 1.2 has two stages. First, for a fixed generic point $x \in \Sigma$ and all radii r below a stopping distance $d_s(x)$, we control the size of projections of $\Sigma \cap B(x, r)$ onto some m -plane $H(r)$ (which may vary as r varies). Here, topology comes into play. To grasp the essence of our idea, it is convenient to think of $\Sigma = M_1 \cup \dots \cup M_N$ as in Example (i) at the beginning of the introduction. For $x \in M_i \setminus \bigcup_{j \neq i} M_j$ and for infinitesimally small radii r we start with the tangent planes $P = T_x M_i$, and note that small $(n - m - 1)$ -spheres that are perpendicular to $T_x M_i$ are nontrivially linked with M_i . Then, for a sequence of growing radii ρ , we rotate P if necessary by a controlled angle to a new position P_ρ in order to keep the projections large. At the same time, we construct a growing connected *excluded region* S_ρ which does not contain any point of Σ in its interior. The size of the projections is controlled via a topological argument, involving the homotopy invariance of the linking number mod 2 of submanifolds. The construction stops at some stopping distance $r = d_s(x)$, and yields another point $y \in \Sigma$ with $|y - x| \approx d_s(x)$ and two smaller balls $B(x, c d_s(x)), B(y, c d_s(x))$, where $c \in (0, 1)$ is an explicit absolute constant, such that

$$\frac{1}{R_{\text{tp}}(z, w)} \gtrsim \frac{1}{d_s(x)}$$

for all $w \in B(y, c d_s(x))$ and a significant proportion of $z \in B(x, c d_s(x))$. In the second stage we use the energy bounds to show that $d(\Sigma) := \inf_{x \in \Sigma} d_s(x)$ is positive and satisfies $d(\Sigma) \geq R_1$, where R_1 is the uniform constant given in Theorem 1.2. The details of that part are given in Section 4; see Lemma 4.2, Lemma 4.3 and their corollaries.

Sections 2 and 3 contain all the necessary prerequisites and are included for the sake of completeness. In Section 2 we gather elementary estimates of angles between planes spanned by nearby almost orthogonal bases, and introduce the class of admissible sets whose definition is designed so that the above sketchy idea can be made precise. In Section 3 we explain how the linking number mod 2 can be used for elements of \mathcal{A} , providing specific statements (and short proofs) for sake of further reference.

Once Theorem 1.2 is proved, we use the Hausdorff convergence of excluded regions defined for generic points $x \in \Sigma$ to obtain a corollary which, roughly speaking, ascertains that for *every* $x \in \Sigma$ and

$r \leq R_1$ there is some plane $H = H_{x,r} \in G(n, m)$ such that $\Sigma \cap B(x, r)$ has large projection onto H and is contained either in $B(x, r/2)$ (where, a priori, at this stage of the reasoning, Σ might behave in a pretty wild way) or in a narrow tubular region $B_\delta(x + H)$, for some specific constant $\delta \ll 1$. A use of energy bounds yields now Theorem 1.3, and an iterative argument implies that in fact Σ must locally be a $C^{1,\kappa}$ graph. All this is done in Section 5. Embeddedness of Σ is established here, too.

Finally, in Section 6, we prove Theorem 1.4 and sharpen the Hölder bounds. To this end, we show that if $\Sigma \cap B$ is a graph of $f \in C^{1,\kappa}$, then ∇f satisfies an improved estimate,

$$|\nabla f(a_1) - \nabla f(a_2)| \leq 2\Phi^*(|a_1 - a_2|/N) + CE^{1/q}|a_1 - a_2|^\mu, \quad (1.8)$$

where $\Phi^*(s)$ stands for the supremum of oscillations of ∇f over all possible balls of radius s , and E is the portion of energy coming from some ball containing a_1, a_2 . The point is that (1.8) holds for some $N = N(q) \gg 1$, so that for $f \in C^{1,\kappa}$ the first term of the right hand side can be viewed as an unimportant, small scale perturbation. The main idea behind (1.8) is that when the integral average of $(1/R_{\text{tp}})^q$ is bounded by K , then there are numerous points u_i in small balls around the a_i , $i = 1, 2$, where $(1/R_{\text{tp}})^q \lesssim K$. A geometric argument implies that for such points $|\nabla f(u_1) - \nabla f(u_2)|$ can be controlled by the second term in the right hand side of (1.8), and a routine iterative reasoning, with a certain Morrey–Campanato flavour, allows us to get rid of the $2\Phi^*$ and finish the whole proof.

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2 Bases, projections, angle estimates, and the class of admissible sets

2.1 Balls, slabs, planes

We write $B(x, r)$ to denote the *closed* ball in \mathbb{R}^n , with center x and radius $r > 0$. The volume of the unit ball in \mathbb{R}^k is denoted by $\omega(k)$.

For a closed set F in \mathbb{R}^n we set

$$U_\delta(F) := \{x \in \mathbb{R}^n : \text{dist}(x, F) < \delta\}, \quad \delta > 0.$$

$G(n, m)$ denotes the Grassmannian of all m -dimensional linear subspaces of \mathbb{R}^n . If $P \in G(n, m)$, then π_P denotes the orthogonal projection of \mathbb{R}^n onto P , and Q_P is the orthogonal projection onto $P^\perp \in G(n, n - m)$.

For two planes $P_1, P_2 \in G(n, m)$ we define their distance (or angle)

$$\angle(P_1, P_2) \equiv d(P_1, P_2) := \|\pi_{P_1} - \pi_{P_2}\|,$$

where the right hand side is the usual norm of the linear map $\pi_{P_1} - \pi_{P_2} : \mathbb{R}^n \rightarrow \mathbb{R}^n$. The Grassmannian $G(n, m)$ equipped with this metric is compact.

Finally, we use the following variant of P. Jones' beta-numbers (see David's and Semmes' monograph [8, Chapter 1, Sec. 1.3] for a discussion):

$$\beta_\Sigma(x, r) := \inf_{L \in G(n, m)} \left(\sup_{y \in \Sigma \cap B(x, r)} \frac{\text{dist}(y, x + L)}{r} \right), \quad x \in \Sigma, \quad r > 0. \quad (2.1)$$

2.2 Nearby planes: bases, projections, angle estimates

Throughout most of the paper, we shall work with estimates of various geometric quantities related to two planes in $G(n, m)$ that form a small angle. For sake of further reference, we gather here several such estimates. We also fix specific constants (which in all cases are far from being optimal) that are needed later, in more involved computations in Sections 4–6. All proofs are elementary, but we provide them to make the exposition complete.

Lemma 2.1. *Assume that $a, b > 0$ and a sequence of nonnegative numbers s_k satisfies $s_1 \leq 1$,*

$$s_{k+1} \leq ak + b \sum_{j=1}^k s_j, \quad k \geq 1.$$

Then for each $A \geq 1 + \max(2a, 2b)$ we have $s_k < A^k$, $k = 1, 2, \dots$

Proof. One proceeds by induction. Clearly, for $k = 1$ we just need $s_1 \leq 1 < A$. For each $A > 1$ the recursive condition for s_{k+1} yields, under the inductive hypothesis,

$$s_{k+1} < ak + \frac{Ab}{A-1}(A^k - 1) \quad (2.2)$$

Now, $A \geq 1 + \max(2a, 2b)$ guarantees that $2ak < (1 + 2a)^k \leq A^k \leq A^{k+1}$ and $\frac{b}{A-1} \leq \frac{1}{2}$. Thus, (2.2) yields $2s_{k+1} < A^{k+1} + A^{k+1} - A < 2A^{k+1}$. \square

Lemma 2.2. *If $X, Y \in G(n, l)$ have orthonormal bases $(e_j) \subset X$ and $(f_j) \subset Y$ such that $|e_j - f_j| \leq \alpha$ for each $j = 1, \dots, l$, then $\angle(X, Y) \leq 2l\alpha$.*

Proof. Take an arbitrary unit vector $v \in \mathbb{R}^n$ and estimate $|\pi_X(v) - \pi_Y(v)|$, expressing both projections in orthonormal bases (e_j) and (f_j) . \square

Lemma 2.3. *Assume that $1 \leq l \leq m \leq n$. If e_1, \dots, e_l is an orthonormal basis of a subspace $X \in G(n, l)$ and $h_1, \dots, h_l \in \mathbb{R}^n$ satisfy $|h_i - e_i| < \varepsilon < \varepsilon_1 := 10^{-1}(10^m + 1)^{-1}$, then $(h_i)_{i=1, \dots, l}$ are linearly independent. Moreover, the Gram-Schmidt orthogonalization process*

$$u_i := \frac{v_i}{|v_i|}, \quad \text{where } v_1 = h_1, \quad v_{k+1} = h_{k+1} - \sum_{j=1}^k \frac{\langle h_{k+1}, v_j \rangle}{|v_j|^2} v_j, \quad k+1 \leq l,$$

yields vectors v_i, u_i ($i = 1, \dots, l$) that satisfy

$$|v_k - h_k| < 10^k \varepsilon, \quad ||v_k| - 1| < (10^k + 1)\varepsilon < \frac{1}{10} \quad \text{for all } k = 1, \dots, l, \quad (2.3)$$

$$|u_k - e_k| < c_1 \varepsilon < \frac{1}{2} \quad \text{for all } i = 1, \dots, l, \quad (2.4)$$

where $c_1 := 2(10^m + 1)$. If $Y = \text{span}(h_1, \dots, h_l)$, then

$$\angle(X, Y) \leq c_2 \varepsilon, \quad (2.5)$$

with $c_2 := 2mc_1 = 4m(10^m + 1)$.

Proof. As $|h_j - e_j| < \varepsilon$ for all j , we have $|\langle h_i, h_j \rangle - \langle e_i, e_j \rangle| < 3\varepsilon$. Therefore, $|\langle h_{k+1}, v_j \rangle| < 3\varepsilon + (1 + \varepsilon)|h_j - v_j|$ for $j = 1, \dots, k$ and $k \leq l - 1$. Using this observation, one proves (2.3) by induction; assuming (2.3) for k and all $j < k$, we obtain

$$\begin{aligned} |v_{k+1} - h_{k+1}| &\leq \sum_{j=1}^k \frac{|\langle h_{k+1}, v_j \rangle|}{|v_j|} < \sum_{j=1}^k \frac{3\varepsilon + (1 + \varepsilon)|v_j - h_j|}{|v_j|} \\ &< \frac{10}{9}3k\varepsilon + \frac{11}{9} \sum_{j=1}^k 10^j \varepsilon < 10^{k+1} \varepsilon, \end{aligned}$$

where the last inequality follows from elementary computations (the estimate is not sharp). This yields the first part of (2.3) for $k + 1$; the second one follows from the triangle inequality.

In particular, we also have $\text{dist}(h_{k+1}, \text{span}(h_1, \dots, h_k)) = |v_{k+1}| > 0$, and therefore h_1, \dots, h_l are linearly independent.

Setting $u_i := v_i/|v_i|$, we easily conclude the proof of the whole lemma. (To check inequality (2.5), apply Lemma 2.2 and note that $l \leq m$.) \square

Lemma 2.4. *Let ε_1 be the constant defined in Lemma 2.3 above. Assume that*

- (i) *there exist orthonormal $e_1, \dots, e_m \in \mathbb{R}^n$ such that $h_i \in B^n(e_i, \delta)$ for $i = 1, \dots, m$, and $\delta < \varepsilon_1/2$;*
- (ii) *$w_i \in B^n(h_i, \varepsilon)$ for all $i = 1, \dots, m$, and $\varepsilon < \varepsilon_1/2$.*

Then the subspaces $H = \text{span}(h_1, \dots, h_m)$ and $W = \text{span}(w_1, \dots, w_m)$ belong to $G(n, m)$, and we have $\angle(H, W) \leq c_3 \varepsilon$ with $c_3 = 14m \cdot 20^m$.

Proof. It follows from Lemma 2.3 that $\dim H = \dim W = m$. We use again the Gram-Schmidt algorithm and set $v_1 = h_1, u_1 = w_1$,

$$v_{k+1} = h_{k+1} - \sum_{j=1}^k \frac{\langle h_{k+1}, v_j \rangle}{|v_j|^2} v_j, \quad u_{k+1} = w_{k+1} - \sum_{j=1}^k \frac{\langle w_{k+1}, u_j \rangle}{|u_j|^2} u_j, \quad k + 1 \leq m.$$

Then, v_i and u_i form orthogonal bases of H and W , respectively. Inequality (2.3) yields $t^{-1} < |u_i|, |v_i| < t$ with $t = 10/9$. We now show that $s_i = \varepsilon^{-1}|u_i - v_i|$ satisfies the assumptions of Lemma 2.1 with $a = 1$ and $b = 8$. For $k = 1$ we have $s_1 = \varepsilon^{-1}|h_1 - w_1| < 1$.

Let $\phi(x) = |x|^{-2}x$. For all x, y in the annulus $\{t^{-1} \leq |z| \leq t\}$ we have $|\phi(x)| \leq t$ and, for $x \neq y$,

$$\begin{aligned} |\phi(x) - \phi(y)| &\leq \frac{|x - y|}{|x|^2} + |y| \left| \frac{1}{|x|^2} - \frac{1}{|y|^2} \right| \leq t^2|x - y| + t \left| \int_{|x|}^{|y|} \frac{2}{\tau^3} d\tau \right| \\ &\leq t^2(1 + 2t^2)|x - y| < 5|x - y|, \quad \text{as } t = 10/9. \end{aligned}$$

Thus, since $w_j, u_j, v_j \in \{t^{-1} \leq |z| \leq t\}$, we obtain

$$\begin{aligned} |u_{k+1} - v_{k+1}| &\leq |h_{k+1} - w_{k+1}| + \left| \sum_{j=1}^k \langle h_j, \phi(v_j) \rangle v_j - \langle w_j, \phi(u_j) \rangle u_j \right| \\ &\leq \varepsilon + \sum_{j=1}^k \left(\varepsilon + t^2(|\phi(v_j) - \phi(u_j)| + |v_j - u_j|) \right) \leq \varepsilon + \sum_{j=1}^k \left(\varepsilon + 6t^2|v_j - u_j| \right). \end{aligned}$$

Hence,

$$s_{k+1} = \varepsilon^{-1} |u_{k+1} - v_{k+1}| \leq (k+1) + b \sum_{j=1}^k s_j$$

for each $b \geq 6t^2$, in particular for $b = 8$. Therefore certainly $s_k \leq 20^k$, $k = 1, \dots, m$, by Lemma 2.1. Keeping in mind that $t^{-1} = \frac{9}{10} \leq |u_j|, |v_j| \leq \frac{10}{9} = t$, we obtain

$$\left| \frac{u_j}{|u_j|} - \frac{v_j}{|v_j|} \right| = ||u_j|\phi(u_j) - |v_j|\phi(v_j)|| < 6t|u_j - v_j| < 7 \cdot 20^m \varepsilon.$$

The inequality $\angle(H, W) \leq c_3 \varepsilon$ follows now from Lemma 2.2. \square

The next two lemmata are concerned with the set

$$S(H_1, H_2) := \{y \in \mathbb{R}^n : \text{dist}(y, H_i) \leq 1 \text{ for } i = 1, 2\}, \quad (2.6)$$

where $H_1 \neq H_2 \in G(n, m)$ form a small angle so that π_{H_1} restricted to H_2 is bijective. Since $\{y \in \mathbb{R}^n : \text{dist}(y, H_i) \leq 1\}$ is convex, closed and centrally symmetric³ for each $i = 1, 2$, we immediately obtain the following:

Lemma 2.5. *$S(H_1, H_2)$ is a convex, closed and centrally symmetric set in \mathbb{R}^n ; $\pi_{H_1}(S(H_1, H_2))$ is a convex, closed and centrally symmetric set in $H_1 \cong \mathbb{R}^m$.*

The next lemma and its corollary provide a key tool for bootstrap estimates in Section 6.

Lemma 2.6. *Let $\varepsilon_1 > 0$ and $c_2 > 0$ denote the constants defined in Lemma 2.3. If $H_1, H_2 \in G(n, m)$ satisfy $0 < \angle(H_1, H_2) = \alpha < \varepsilon_1$, then there exists an $(m-1)$ -dimensional subspace $W \subset H_1$ such that*

$$\pi_{H_1}(S(H_1, H_2)) \subset \{y \in H_1 : \text{dist}(y, W) \leq 5c_2/\alpha\}.$$

Proof. Let $H := H_1 \cap H_2$; we have $k := \dim H < m$. For $i = 1, 2$ set $X_i = \{x \in H_i : x \perp H\}$. Then, H_i is the orthogonal sum of H and X_i . Let $X := X_1 \oplus X_2$; by construction, $X \perp H$. Finally, let L be the orthogonal complement of $H \oplus X = H_1 \oplus H_2$ in \mathbb{R}^n , so that \mathbb{R}^n is equal to $H \oplus X \oplus L$, and the spaces H, X, L are pairwise orthogonal. It is now easy to see, directly by definition, that $\angle(H_1, H_2) = \angle(X_1, X_2)$.

Step 1. We shall first show that there exists a vector $x_1 \in X_1$ such that

$$|x_1| = 5c_2/\alpha, \quad x_1 \notin \pi_{H_1}(S(H_1, H_2)). \quad (2.7)$$

Fix an orthonormal basis e_1, \dots, e_{m-k} of X_1 . Since $\angle(X_1, X_2) = \alpha$, we have $|e_j - \pi_{X_2}(e_j)| \leq \alpha$. Applying Lemma 2.3 with $l = m-k$ to X_1 and X_2 , we check that the $\pi_{X_2}(e_j)$, $j = 1, \dots, m-k$, form a basis of X_2 , and $|e_j - \pi_{X_2}(e_j)| \geq \alpha/c_2$ for at least one $j \in \{1, 2, \dots, m-k\}$. Assume w.l.o.g. that this is the case for $j = 1$. Thus, there are no points of X_2 in the interior of $B := B^n(e_1, \alpha/c_2)$, and therefore there are no points of X_2 in the interior of the cone

$$K := \{y \in \mathbb{R}^n : y = tv, t \in \mathbb{R}, v \in B\}.$$

Set $\lambda := 5c_2/\alpha$. Then $\lambda B = B(\lambda e_1, 5) \subset K$, so that the closed ball $B(\lambda e_1, 4) \subset \text{int } K$. Hence,

$$B(\lambda e_1, 3) \cap \{y \in \mathbb{R}^n : \text{dist}(y, X_2) \leq 1\} = \emptyset. \quad (2.8)$$

³The term *central symmetry* is used here for central symmetry with respect to 0 in \mathbb{R}^n .

Let I denote the segment $\{se_1 : |s - \lambda| \leq 1\}$; we claim that $\pi_{H_1}(S(H_1, H_2)) \cap I = \emptyset$. To check this, we argue by contradiction. If $y \in S(H_1, H_2)$ and $\pi_{H_1}(y) \in I$, then, decomposing $y = h + x + l$ where $h \in H$, $x \in X$, and $l \in L = (H \oplus X)^\perp$, we have

$$\pi_{H_1}(y) = h + \pi_{H_1}(x) = h + \pi_{X_1}(x) = se_1$$

for some s , $|s - \lambda| \leq 1$. As $H \perp X$ and $e_1 \in X_1 \subset X$, this yields $h = 0$ and $y = se_1 + \beta$ for some $\beta \perp X_1$. Now, (2.8) shows that if $y \in S(H_1, H_2)$, then we must have $|\beta|^2 \geq 3^2 - 1^2 = 8$. This, however, yields $\text{dist}(y, X_1) = |\beta| > 2$, which contradicts the assumption $y \in S(H_1, H_2)$. Thus, $x_1 := \lambda e_1 = (5c_2/\alpha)e_1$ satisfies (2.7).

Now, in order to prove the existence of the desired subspace $W \subset H_1$, consider the function

$$w \mapsto g(w) := \inf\{t > 0 : tw \notin \pi_{H_1}(S(H_1, H_2))\} \in \mathbb{R}_+ \cup \{\infty\}$$

defined on the unit sphere in H_1 . If $w \in H$, then $g(w) = \infty$. Since $B^n(0, 1) \subset S(H_1, H_2)$, we have $g \geq 1$ everywhere. Note that if $g(w) = s$ then $sw \in \pi_{H_1}(S(H_1, H_2))$ and $tw \notin \pi_{H_1}(S(H_1, H_2))$ for every $t > s$. (Thus, $g(w)$ is the ‘exit time’ that we need to leave $\pi_{H_1}(S(H_1, H_2))$, travelling with unit speed from 0 in the direction given by w .)

Step 2. We shall first show that there exists a vector $w_0 \in H_1$, $|w_0| = 1$, such that

$$g(w_0) = r = \inf g < \frac{5c_2}{\alpha}.$$

Since, by Step 1, we have $g(e_1) < \lambda = 5c_2/\alpha$, it is of course enough to show that $1 \leq r = \inf g$ is achieved on the unit sphere of H_1 . Take a sequence of unit vectors $w_i \in H_1$ such that $g(w_i) \rightarrow \inf g$; passing to a subsequence, we can assume $w_i \rightarrow w_0$ as $i \rightarrow \infty$. Suppose now that $g(w_0) > \inf g$. Then, for some fixed $\varepsilon > 0$ we have $g(w_0) > g(w_i) + \varepsilon > r = \inf g$ for all $i \gg 1$. Consider the points $p_0 = g(w_0)w_0$ and $p_i = g(w_i)w_i$ in $\pi_{H_1}(S(H_1, H_2))$. Then, $\frac{p_i - p_0}{|p_i - p_0|} \rightarrow -w_0$ as $i \rightarrow \infty$. By definition of r and convexity,

$$\pi_{H_1}(S(H_1, H_2)) \supset \text{conv}\left(\{p_0\} \cup (B^n(0, r) \cap H_1)\right).$$

Thus, for all i such that $g(w_i) < r + (\varepsilon/2)$ and $\angle(w_i, w_0) < \arccos(r/(r + \varepsilon)) - \arccos(r/(r + \frac{\varepsilon}{2}))$ the point p_i is in the interior of $\text{conv}\left(\{p_0\} \cup (B^n(0, r) \cap H_1)\right)$. Then, however, by definition of g we obtain $g(w_i) > |p_i| = g(w_i)$, a contradiction which shows that $g(w_0) = \inf g$.

Step 3. $W = \{y \in H_1 : y \perp w_0\}$ satisfies the desired condition. Note that W is chosen so that the set $F := \{y \in H_1 : \text{dist}(y, W) \leq r = \inf g\}$ is the ‘narrowest strip in H_1 ’ containing $\pi_{H_1}(S(H_1, H_2))$.

Indeed, if there was a point $y \in \pi_{H_1}(S(H_1, H_2)) \setminus F$, then, taking the straight line through y and $y_0 = rw_0 \in \partial F \cap \pi_{H_1}(S(H_1, H_2))$ (with $r = \inf g$), one could easily reach a contradiction: take a unit vector v in $\text{span}(y, y_0)$, $v \perp y - y_0$, and use convexity of $\pi_{H_1}(S(H_1, H_2))$ to show that then $g(v) < r = g(w_0) = \inf g$ (for otherwise the straight segment connecting the points y and $g(v)v$ contained in $\pi_{H_1}(S(H_1, H_2))$ would intersect the ray $\{y_0 + tw_0 : t > 0\}$ contradicting the definition of $g(w_0)$).

This completes the proof of Lemma 2.6. \square

The next Lemma is practically obvious.

Lemma 2.7. Suppose that $H \in G(n, m)$ and a set $S' \subset H$ is contained in $\{y \in H : \text{dist}(y, W) \leq d\}$ for some $d > 0$, where W is an $(m - 1)$ -dimensional subspace of H . Then

$$\mathcal{H}^m(S' \cap B^n(a, s)) \leq 2^m s^{m-1} d$$

for each $a \in H$ and each $s > 0$.

Proof. Decomposing each $y \in S' \cap B^n(a, s)$ as $y = \pi_W(y) + (y - \pi_W(y))$, one sees that $S' \cap B^n(a, s)$ is contained in a rectangular box with $(m-1)$ sides parallel to W and of length $2s$ and the remaining side perpendicular to W and of length $2d$. \square

Lemma 2.8. *If two planes $H_1, H_2 \in G(n, m)$ satisfy $\angle(H_1, H_2) \leq \varepsilon < m^{-1}2^{-m}$, then*

$$\mathcal{H}^m(\pi_{H_1}(A)) \geq (1 - m\varepsilon 2^m) \mathcal{H}^m(A) \quad (2.9)$$

for every \mathcal{H}^m -measurable set $A \subset H_2$.

Proof. It is enough to prove the inequality when A is the m -dimensional unit cube in H_2 , and $\mathcal{H}^m(A) = 1$. Fix an orthonormal basis e_1, \dots, e_m of H_2 and let $f_i := \pi_{H_1}(e_i)$ for $i = 1, \dots, m$. Then, by Hadamard's inequality,

$$\begin{aligned} \mathcal{H}^m(\pi_{H_1}(A)) &= |f_1 \wedge \dots \wedge f_m| \\ &\geq |e_1 \wedge \dots \wedge e_m| - \sum_{j=1}^m |f_j - e_j| \prod_{j < i \leq m} (1 + |f_i - e_i|) \\ &\geq 1 - \varepsilon \sum_{j=1}^m (1 + \varepsilon)^{m-j} \geq 1 - m\varepsilon 2^m. \end{aligned}$$

2.3 The class of admissible sets

Let us now give a precise definition of the class of admissible surfaces. Intuitively speaking, the energy functional \mathcal{E}_q can be defined for all $\Sigma \subset \mathbb{R}^n$ compact, $\mathcal{H}^m(\Sigma) < \infty$, which are a union of continuous images of m -dimensional closed manifolds of class C^1 , satisfying two additional conditions. One of them ensures that Σ is pretty flat near \mathcal{H}^m -almost all its points x , so that we have, in a sense, a ‘mock’ tangent plane H_x to Σ at x . A priori, H_x does not even have to coincide with the classic tangent plane. The second condition guarantees, as we shall see later, that small $(n-m-1)$ -dimensional spheres centered at x and parallel to $(H_x)^\perp$ are nontrivially linked with the surface Σ .

As we have already said in the introduction, it might be convenient to think of the following example. Assume that $\Sigma_1, \dots, \Sigma_N$ are embedded, compact, closed m -dimensional C^1 -submanifolds of \mathbb{R}^n . They might intersect each other but only along sets of m -dimensional measure zero, so that $\mathcal{H}^m(\Sigma_i \cap \Sigma_j) = 0$ whenever $i \neq j$. Then, for any bilipschitz homeomorphism $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$,

$$\Sigma := f(\Sigma_1 \cup \dots \cup \Sigma_N)$$

is an admissible surface.

The definition of admissible surfaces involves the notion of degree modulo 2; here are its relevant properties.

Theorem 2.9 (Degree modulo 2). *Let M, N be compact manifolds of class C^1 without boundary and of the same dimension k . Assume that N is connected. There exists a unique map*

$$\deg_2: C^0(M, N) \rightarrow \mathbb{Z}_2 = \{0, 1\}$$

such that:

- (i) *If $\deg_2 g = 1$, then $g \in C^0(M, N)$ is surjective;*

- (ii) If $H: M \times [0, 1] \rightarrow N$ is continuous, $f = H(\cdot, 0)$ and $g = H(\cdot, 1)$, then $\deg_2 f = \deg_2 g$;
- (iii) If $f: M \rightarrow N$ is of class C^1 and $y \in N$ is an arbitrary regular value of f , then

$$\deg_2 f = \#f^{-1}(y) \pmod{2}.$$

For a proof, see e.g. the monograph of M.W. Hirsch [15, Chapter 5], Theorem 1.6 and the surrounding comments. Blatt gives a detailed presentation of degree modulo 2 (even for noncompact manifolds) in his thesis [4].

Now, let $\delta \in (0, 1)$ and let I be a finite or countable set of indices.

Definition 2.10. We say that a compact set $\Sigma \subset \mathbb{R}^n$ is an *admissible* (m -dimensional) *surface of class* $\mathcal{A}(\delta)$ if the following conditions are satisfied.

(H1) Ahlfors regularity. $\mathcal{H}^m(\Sigma) < \infty$ and there exists a constant $K = K_\Sigma$ such that

$$\mathcal{H}^m(\Sigma \cap B^n(x, r)) \geq K_\Sigma r^m \quad \text{for all } x \in \Sigma, 0 < r \leq \text{diam } \Sigma. \quad (2.10)$$

(H2) Structure. There exist compact, closed m -dimensional manifolds M_i of class C^1 and continuous maps $f_i: M_i \rightarrow \mathbb{R}^n$, $i \in I$, where I is at most countable, such that $\Sigma = \bigcup_{i \in I} f_i(M_i) \cup Z$, where $\mathcal{H}^m(Z) = 0$.

(H3) Mock tangent planes and δ -flatness. There exists a dense subset $\Sigma^* \subset \Sigma$ with the following property: $\mathcal{H}^m(\Sigma \setminus \Sigma^*) = 0$ and for each $x \in \Sigma^*$ there is an m -dimensional plane $H = H_x \in G(n, m)$ and a radius $r_0 = r_0(x) > 0$ such that

$$|y - x - \pi_H(y - x)| < \delta |y - x| \quad \text{for each } y \in B^n(x, r_0) \cap \Sigma, y \neq x. \quad (2.11)$$

(H4) Linking. If $x \in \Sigma^*$ and $r_0(x)$ is given by (H3) above, then there exists an $i \in I$ such that the map⁴

$$\Phi_i: M_i \times \mathbb{S}^{n-m-1}(x, r_0(x); (H_x)^\perp) \ni (w, z) \mapsto \frac{f_i(w) - z}{|f_i(w) - z|} \in \mathbb{S}^{n-1}$$

satisfies the condition $\deg_2 \Phi_i = 1$. (Here we use the notation $\mathbb{S}^l(\xi, \rho; P) := \xi + \{v \in P: |v| = \rho\}$ for $\xi \in \mathbb{R}^n$, $\rho > 0$ and $P \in G(n, l)$.)

Example 2.11. If Σ is a compact, connected manifold of class C^1 without boundary, embedded in \mathbb{R}^n , then $\Sigma \in \mathcal{A}(\delta)$ for every $\delta \in (0, 1)$.

We can take $Z = \emptyset$, $I = \{1\}$, $f_1 = \text{id}_{\mathbb{R}^n}$, and $\Sigma^* = \Sigma$; (H1) and (H2) follow. It is clear that Condition (H3) is satisfied if we choose $H_x = T_x \Sigma$ for $x \in \Sigma$. Condition (H4) is then satisfied, too. In this simple model case (H4) ascertains that small $(n - m - 1)$ -dimensional spheres centered at the points of an embedded manifold Σ and contained in planes that are normal to Σ are linked with that manifold; see e.g. [23, pp. 194-195] for the definition of linking coefficient. (We do not assume orientability of M_i ; this is why degree modulo 2 is used.)

Note that if $\delta > 0$ is fixed, then we are not forced to set $H_x \equiv T_x \Sigma$; conditions (H3) and (H4) in this example would be satisfied also if H_x were sufficiently close to $T_x \Sigma$. Thus, for given Σ satisfying (H1) and (H2) the choice of H_x does not have to be unique.

⁴Note that Φ_i is well defined, as $f_i(w) \in \Sigma$, and $z \notin \Sigma$ by virtue of (H3).

The next two examples show that we can allow Σ to have several C^1 -pieces that intersect along sets of m -dimensional measure zero, and are embedded away from those sets.

Example 2.12. If Σ is connected, $\Sigma = \bigcup_{i=1}^N \Sigma_i$, where Σ_i are compact, connected manifolds of class C^1 without boundary, embedded in \mathbb{R}^n , and moreover

$$\mathcal{H}^m(\Sigma_i \cap \Sigma_j) = 0 \quad \text{for } i \neq j,$$

then $\Sigma \in \mathcal{A}(\delta)$ for every $\delta \in (0, 1)$.

The set I is now equal to $\{1, \dots, N\}$ and we set

$$\Sigma^* := \Sigma \setminus S, \quad S := \bigcup_{1 \leq i < j \leq N} (\Sigma_i \cap \Sigma_j); \quad (2.12)$$

for each $x \in \Sigma^*$ there is a unique i such that $x \in \Sigma_i$ and we take $H_x := T_x \Sigma_i$. Conditions (H1) and (H2) are clearly satisfied with $Z = \emptyset$ and $f_i = \text{id}_{\mathbb{R}^n}$ for $i = 1, \dots, N$, and the verification of (H3) and (H4) is similar to the previous example; one just has to ensure that for $x \in \Sigma^* \cap \Sigma_i$ the radius $r_0 = r_0(x)$ is chosen so that $r_0 < \text{dist}(x, S)$.

Example 2.13. Let the M_i , $i \in I = \{1, \dots, N\}$, be compact, connected m -dimensional C^1 -manifolds without boundary. Let $f_i: M_i \rightarrow \mathbb{R}^n$ be C^1 -immersions, and let $\Sigma_i = f_i(M_i)$ for $i = 1, \dots, N$. If $\Sigma = \bigcup \Sigma_i$ is connected,

$$\mathcal{H}^m(\Sigma_i \cap \Sigma_j) = 0 \quad \text{for } i \neq j,$$

and

$$\mathcal{H}^m(\{y \in \Sigma_i : \#f_i^{-1}(y) > 1\}) = 0 \quad \text{for all } i = 1, \dots, N,$$

then $\Sigma \in \mathcal{A}(\delta)$ for every $\delta \in (0, 1)$. We leave the verification to the reader.

It is also clear that the condition that all maps f_i in the previous example be of class C^1 is too strong. We can allow $\Sigma_i = f_i(M_i)$ to have large intersections with other Σ_j as long as the flatness condition in (H3) is satisfied, and we need H_x only for a.e. $x \in \Sigma$. Thus, it is relatively easy to give more examples of admissible surfaces.

Example 2.14. If $h: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a bilipschitz homeomorphism, and we take Σ as in Example 2.12, then

$$\tilde{\Sigma} \equiv h(\Sigma) \subset \bigcap_{\delta \in (0,1)} \mathcal{A}(\delta).$$

Indeed, we then set $f_i = h \circ \text{id}_{\Sigma_i}$. Let S be given by (2.12). To define $\tilde{\Sigma}^*$, we use compactness and smoothness of the Σ_i to fix a radius $r > 0$ with the following property: for each $i = 1, 2, \dots, N$ and each point $a \in \Sigma_i$ there is a function $g_a: P_a \rightarrow P_a^\perp$, $P_a = T_a \Sigma_i \in G(n, m)$, such that $|Dg_a| \leq 1$ and

$$\Sigma_i \cap B^n(a, r) = \text{graph } g_a \cap B^n(a, r).$$

We also let $G_a(\xi) = (\xi, g_a(\xi))$ for $\xi \in P_a$. Now, a point $y \in h(\Sigma)$ is in $\tilde{\Sigma}^*$ if $y \notin h(S)$ (we exclude the intersections), and moreover there is $i \in \{1, \dots, N\}$ and an $a \in \Sigma_i$ such that $y = h(\xi, g_a(\xi))$ for some $\xi \in P_a \cap B^n(a, r)$ which is a point where $F := h \circ G_a: P_a \rightarrow \mathbb{R}^n$ is differentiable.

It follows from Rademacher's theorem that $\tilde{\Sigma}^*$ has full measure and is dense in $\tilde{\Sigma} = h(\Sigma)$.

Condition (H1) is also satisfied, since bilipschitz maps distort the measure \mathcal{H}^m at most by a constant factor. To check (H3), one notes that as $F = h \circ G_a: P_a \rightarrow \mathbb{R}^n$ is bilipschitz, its differential DF must have maximal rank m at all points where it exists; it is then a simple exercise to check that for $y = h(x) \in \tilde{\Sigma}^*$ the plane $H_y = DF(x)(P_a)$ satisfies all requirements of Condition (H3) for all $\delta \in (0, 1)$. To check (H4), one can use the homotopy invariance of the degree; we leave the details to the reader.

We do not have a simple characterization of the class of admissible surfaces. However, it contains weird countably rectifiable sets, too.

Example 2.15 (Stacks of spheres or cubes). (a) For $i = 0, 1, 2, \dots$ let $p_i = (2^{-i}, 0, 0) \in \mathbb{R}^3$, $c_i = (p_i + p_{i+1})/2$, $r_i = 2^{-i-2} > 0$, $M_i = \Sigma_i = \mathbb{S}^2(c_i, r_i) \subset \mathbb{R}^3$ (so that the spheres Σ_i and Σ_{i+1} touch each other at p_{i+1}), and let $f_i = \text{id}_{M_i}$. Set $\Sigma = \bigcup_{i=0}^{\infty} \Sigma_i \cup \{0\}$. Then, Σ is an admissible surface, belonging to $\mathcal{A}(\delta)$ for each $\delta > 0$. All points of Σ except 0 and the p_i for $i \geq 1$ belong to Σ^* . For $x \in \Sigma^*$, one verifies (H3) and (H4) just as in Example 2.12. Moreover, (H1) is also valid. To see this, fix $x \in \Sigma \setminus \{0\}$. If $x \in \Sigma_i$ and $r \leq 2r_i$, then

$$\mathcal{H}^2(\Sigma \cap B(x, r)) \geq \mathcal{H}^2(\Sigma_i \cap B(x, r)) = \pi r^2,$$

by the standard formula for the area of a spherical cap. If $r > 2r_i$ but $r \leq \text{diam } \Sigma = 1$, then it is possible to check that the largest of all Σ_j completely contained in $B(x, r) \cap \Sigma$ has $r_j \in [r/6, r/2]$. Estimating $\mathcal{H}^2(\Sigma \cap B(x, r))$ from below by $\mathcal{H}^2(\Sigma_j \cap B(x, r))$, we obtain (H1) for $x \neq 0$; a similar argument works for $x = 0$.

(b) A modification of the above example yields the following (see Figure 1): set

$$\Sigma = \bigcup_{i=0}^{\infty} \left(\bigcup_{k=1}^{2^i} \Sigma_{i,k} \right) \cup Z,$$

where $\Sigma_{i,k}$ is the surface of a cube of side length 2^{-i} , and Z is a segment of length 1. To be more specific, $\Sigma_{0,1} = \partial([0, 1]^3) \subset \mathbb{R}^3$, and we let $\Sigma_{i,k}$ be a translated copy of $2^{-i} \cdot \Sigma_{0,1} = \partial[0, 2^{-i}]^3$,

$$\Sigma_{i,k} := \partial([0, 2^{-i}]^3) + (1 - 2^{-i})(e_1 + 2e_3) + (k - 1)2^{-i}e_2,$$

so that, for fixed i , the $\Sigma_{i,k}$ with $k = 1, \dots, 2^i$ form a layer of touching cubes stacked on top of the union of all the previous $\Sigma_{j,s}$, $0 \leq j < i$ and $1 \leq s \leq 2^j$. Finally, set $Z = \{(1, t, 2) : t \in [0, 1]\}$; we add this segment to the union of all $\Sigma_{i,k}$ to make Σ closed.

It is possible to check that if Σ^* is equal to the union of the interiors of all the faces of the cubes (which is a dense set of full surface measure in Σ), then (H3) and (H4) are satisfied. Ahlfors regularity of Σ can be checked as in (a) above.

Remark 2.16. The mock tangent planes H_x are not unique in the definition of the class $\mathcal{A}(\delta)$ but *a posteriori* it follows from Theorem 1.4 that if $\Sigma \in \mathcal{A}(\delta)$, then for any $x \in \Sigma^*$ there is at most *one* choice of the H_x (up to a set of zero measure) if one wants $\mathcal{E}_q(\Sigma)$ to be finite. Thus, finiteness of the energy is a very strong assumption: it forces us to abandon the apparent freedom of choice of the H_x , and forces Σ to be a single embedded manifold, with a controlled amount of bending at a given length scale, depending only on the energy.

3 Topological prerequisites

To guarantee the existence of big projections later on, we shall need a topological invariant, which is a version of the linking number modulo 2.

Definition 3.1 (Linking number modulo 2). Assume that Σ is an admissible surface of class $\mathcal{A}(\delta)$ and N^{n-m-1} is a compact, closed $(n - m - 1)$ -dimensional manifold of class C^1 , embedded in \mathbb{R}^n and such that $N^{n-m-1} \cap \Sigma = \emptyset$.

For each $i \in I$ and for the manifolds M_i which satisfy (H2) and (H4) of Definition 2.10, let

$$G_i: M_i \times N^{n-m-1} \ni (w, z) \mapsto \frac{f_i(w) - z}{|f_i(w) - z|} \in \mathbb{S}^{n-1}. \quad (3.1)$$

We set

$$\text{lk}_2(\Sigma^m, N^{n-m-1}) := \begin{cases} 1 & \text{if } \deg_2 G_i = 1 \text{ for some } i \in I, \\ 0 & \text{if } \deg_2 G_i = 0 \text{ for all } i \in I. \end{cases}$$

We shall use this definition mostly in the case where N^{n-m-1} is a round sphere (or an ellipsoid with ratio of axes very close to 1) contained in some $(n-m)$ -affine plane in \mathbb{R}^n .

We need the following four properties of this invariant.

Lemma 3.2 (Homotopy invariance). *Let $\Sigma \in \mathcal{A}(\delta)$ and let N be a compact, closed $(n-m-1)$ -dimensional manifold of class C^1 , and let $N_j := h_j(N)$ for $j = 0, 1$, where h_j is a C^1 embedding of N into \mathbb{R}^n such that $N_j \cap \Sigma = \emptyset$. If there is a homotopy*

$$H: N \times [0, 1] \rightarrow \mathbb{R}^n \setminus \Sigma$$

such that $H(\cdot, 0) = h_0$ and $H(\cdot, 1) = h_1$, then

$$\text{lk}_2(\Sigma, N_0) = \text{lk}_2(\Sigma, N_1).$$

Proof. Note that the mappings

$$g_{i,j}: M_i \times N \ni (w, z) \mapsto \frac{f_i(w) - h_j(z)}{|f_i(w) - h_j(z)|} \in \mathbb{S}^{n-1}, \quad i \in I, \quad j = 0, 1,$$

are such that $g_{i,0}$ is homotopic to $g_{i,1}$ for each $i \in I$. Thus, the lemma follows directly from Theorem 2.9 (ii). \square

Lemma 3.3 (Small spheres in ‘mock’ normal planes are linked with Σ). *Assume that $\Sigma \in \mathcal{A}(\delta)$ and $x \in \Sigma^*$. Then for all $r \in (0, r_0(x))$ and for $V_x = (H_x)^\perp$ we have*

$$\text{lk}_2(\Sigma, \mathbb{S}^{n-m-1}(x, r; V_x)) = 1, \quad (3.2)$$

where $r_0(x)$ is the constant in Condition (H3) of Definition 2.10.

Proof. Due to condition (H3), each sphere $\mathbb{S}^{n-m-1}(x, r; V_x)$ with $r \in (0, r_0(x))$ can be deformed homotopically to $\mathbb{S}^{n-m-1}(x, r_0(x); V_x)$; we simply adjust the radius, changing it linearly. Since the image of that homotopy is disjoint from Σ , the lemma follows from (H4) and Lemma 3.2. \square

Lemma 3.4 (Distant spheres are not linked). *If $\Sigma \in \mathcal{A}(\delta)$, $0 < \varepsilon < r < 2\varepsilon$ and $\text{dist}(y, \Sigma) > 3\varepsilon$, then*

$$\text{lk}_2(\Sigma, \mathbb{S}^{n-m-1}(y, r; V)) = 0$$

for each plane $V \in G(n, n-m)$.

Proof. Fix an arbitrary $i \in I$. Set $N^{n-m-1} \equiv \mathbb{S}^{n-m-1}(y, r; V)$ and let $G_i: M_i \times N^{n-m-1} \rightarrow \mathbb{S}^{n-1}$ be defined by (3.1). We shall prove that $\deg_2 G_i = 0$. To this end, consider the homotopy

$$H: M_i \times N^{n-m-1} \times [0, 1] \rightarrow \mathbb{S}^{n-1}$$

given by

$$H(w, z, t) = \frac{f_i(w) - (y + (1-t)(z-y))}{|f_i(w) - (y + (1-t)(z-y))|}, \quad w \in M_i, z \in N^{n-m-1}, t \in [0, 1]. \quad (3.3)$$

It is easy to see that H is well defined and continuous; we have $H(w, z, 0) = G_i(w, z)$. Thus, by Theorem 2.9 (ii), $\deg_2 G_i = \deg_2 H(\cdot, \cdot, t)$ for each $t \in (0, 1]$.

If $m < n - 1$, then the image of $H(\cdot, \cdot, 1)$ in \mathbb{S}^{n-1} is the same as image of Σ_i under the map $\xi \mapsto (\xi - y)/|\xi - y|$ which is Lipschitz in a neighbourhood of Σ_i . Since $\mathcal{H}^m(\Sigma_i) < \infty$, $H(\cdot, \cdot, 1)$ cannot be surjective, since the \mathcal{H}^{n-1} -measure of its image is zero. Thus, we obtain $\deg_2 H(\cdot, \cdot, 1) = 0 = \deg_2 H(\cdot, \cdot, 0) = \deg_2 G_i$.

If $m = n - 1$, we first approximate f_i by a smooth map $\tilde{f}_i: M_i \rightarrow \mathbb{R}^n$, so that $\|f_i - \tilde{f}_i\|_\infty < \varepsilon/2$. Then,

$$\tilde{G}_i(w, z) := (\tilde{f}_i(w) - z)/|\tilde{f}_i(w) - z|, \quad (w, z) \in M_i \times N^{n-m-1},$$

satisfies⁵ $\deg_2 G_i = \deg_2 \tilde{G}_i$. Next, we define \tilde{H} by (3.3) with f_i 's replaced by \tilde{f}_i 's. If $\tilde{H}(\cdot, \cdot, 1)$ has no regular points (= points where the differential has rank equal to $n - 1$), then its Jacobian is zero, and $\tilde{H}(\cdot, \cdot, 1)$ is not surjective. If $\tilde{H}(\cdot, \cdot, 1)$ has at least one regular point, then since N^{n-m-1} consists of two distinct points z_1, z_2 and $\tilde{H}(w, z_1, 1) = \tilde{H}(w, z_2, 1)$, we see each regular value of $\tilde{H}(\cdot, \cdot, 1)$ has an even number of preimages in $M_i \times N^{n-m-1}$. Hence, in either case $\deg_2 \tilde{H}(\cdot, \cdot, 1) = 0 = \deg_2 \tilde{H}(\cdot, \cdot, 0) = \deg_2 \tilde{G}_i = \deg_2 G_i$.

Lemma 3.5. *If $\Sigma \in \mathcal{A}(\delta)$ and for some $y \in \mathbb{R}^n$, $r > 0$ and $V \in G(n, n - m)$ we have*

$$\text{lk}_2(\Sigma, \mathbb{S}^{n-m-1}(y, r; V)) = 1$$

then the disk

$$D^{n-m}(y, r; V) := y + \{v \in V : |v| \leq r\}$$

contains at least one point of Σ .

Proof. Suppose this were not the case. Then $\text{dist}(\Sigma, D^{n-m}(y, r; V)) > 3\varepsilon$ for some $\varepsilon > 0$. We deform continuously the sphere $\mathbb{S}^{n-m-1}(y, r; V)$ to $\mathbb{S}^{n-m-1}(y, 3\varepsilon/2; V)$, staying all the time in $y + V$, at the distance at least 3ε to Σ . This yields

$$\text{lk}_2(\Sigma, \mathbb{S}^{n-m-1}(y, r; V)) = \text{lk}_2(\Sigma, \mathbb{S}^{n-m-1}(y, 3\varepsilon/2; V)) = 0$$

by Lemma 3.2 and Lemma 3.4, a contradiction. \square

4 Uniform Ahlfors regularity

4.1 Good couples of points

We introduce here the notion of a *good couple*. It expresses in a quantitative way the following rough idea: if there are two points $x, y \in \Sigma$ such that the distance from y to a substantial portion of the affine planes $z + H_z$ (where z is very close to x) is comparable to $|x - y|$, then a certain portion of energy comes *only from the neighbourhood of points forming such a configuration*. Quantifying this, and iterating the resulting information in the next section, we eventually are able to pinpoint some of the local and global properties of the surface.

Recall that Q_{H_z} stands for the orthogonal projection onto $(H_z)^\perp$.

⁵Just move $f_i(w)$ to $\tilde{f}_i(w)$ along a segment, which avoids N^{n-m-1} , as $\|f_i - \tilde{f}_i\|_\infty < \varepsilon/2$ and $\text{dist}(\tilde{f}_i(w), N) > \varepsilon/2$.

Definition 4.1 (Good couples). *We say that $(x, y) \in \Sigma \times \Sigma$ is a (λ, α, d) -good couple if and only if the following two conditions are satisfied:*

(i) $d/2 \leq |x - y| \leq 2d$;

(ii) *The set*

$$S(x, y; \alpha, d) := \{z \in B^n(x, \alpha^2 d) \cap \Sigma^* : |Q_{H_z}(y - z)| \geq \alpha d\}$$

satisfies

$$\mathcal{H}^m(S(x, y; \alpha, d)) \geq \lambda \mathcal{H}^m(B^m(0, \alpha^2 d)) = \lambda \omega(m) \alpha^{2m} d^m.$$

We shall be using this definition for fixed $0 < \alpha, \lambda \ll 1$ depending only on n and m . Intuitively, good couples force the energy to be large. Once we have a (λ, α, d) -good couple, then $1/R_{\text{tp}}$ must be $\gtrsim \alpha d^{-1}$ on a set in $\Sigma \times \Sigma$ of $\mathcal{H}^m \otimes \mathcal{H}^m$ -measure roughly d^{2m} . Thus, for $q > 2m$, one cannot have d small and $\mathcal{E}_q(\Sigma)$ small simultaneously. We quantify that in Lemma 4.4.

Lemma 4.2. *If $(x, y) \in \Sigma \times \Sigma$ is a (λ, α, d) -good couple with $\alpha < \frac{1}{2}$ and an arbitrary $\lambda \in (0, 1]$, then*

$$\frac{1}{R_{\text{tp}}(z, w)} > \frac{1}{9} \frac{\alpha}{d} \quad (4.1)$$

for all $z \in S(x, y; \alpha, d)$ and $w \in B^n(y, \alpha^2 d)$.

Proof. For z, w as above we have

$$\begin{aligned} |Q_{H_z}(w - z)| &= |Q_{H_z}(y - z) + Q_{H_z}(w - y)| \\ &\geq \alpha d - |w - y| \quad \text{by Def. 4.1 (ii)} \\ &> \frac{\alpha d}{2} \quad \text{as } \alpha < 1/2. \end{aligned}$$

Moreover, $|w - z| \leq |x - y| + |x - z| + |w - y| \leq 2d + 2\alpha^2 d < 3d$. Thus, by (1.2),

$$\frac{1}{R_{\text{tp}}(z, w)} = \frac{2 \text{dist}(w, z + H_z)}{|w - z|^2} = \frac{2|Q_{H_z}(w - z)|}{|w - z|^2} > \frac{\alpha d}{(3d)^2} = \frac{1}{9} \frac{\alpha}{d}.$$

4.2 Finding good couples and large projections

To prove uniform Ahlfors regularity, we shall demonstrate that each Σ with finite energy cannot penetrate certain conical regions of \mathbb{R}^n . The construction of those regions will guarantee that in a neighbourhood of each point $x \in \Sigma^*$ the projections of Σ onto suitably chosen m -planes passing through x are large, and a bound on the energy will allow us to prove that such neighbourhoods have to be uniformly large, independent of the particular point $x \in \Sigma^*$ we have chosen.

For a plane $H \in G(n, m)$ and $\delta \in (0, 1)$ we set

$$C(\delta, H) := \{z \in \mathbb{R}^n : |Q_H(z)| \geq \delta |z|\}, \quad (4.2)$$

$$C_r(\delta, H) := C(\delta, H) \cap B^n(0, r). \quad (4.3)$$

(These are closed ‘double cones’ with ‘axis’ equal to H^\perp . Note that if $n > m + 1$, then the interior of $C(\delta, H)$ and of $C_r(\delta, H)$ is connected.) We shall also use the intersections of cones with annuli,

$$A_{R,r}(x, \delta, W) := x + \text{int} \left(C_R(\delta, W) \setminus B^n(0, r) \right). \quad (4.4)$$

Lemma 4.3 (Stopping distances, good couples and large projections).

There exist constants $\eta = \eta(m)$, $\delta = \delta(m)$, $\lambda = \lambda(n, m) \in (0, \frac{1}{9})$ which depend only on n, m , and have the following property.

For every $\Sigma \in \mathcal{A}(\delta)$ and every $x \in \Sigma^*$ there exist $d \equiv d_s(x) > 0$ and $y \in \Sigma$ such that

- (i) (x, y) is a (λ, η, d) -good couple;
- (ii) for each $r \in (0, d]$ there exists a plane $H(r) \in G(n, m)$ such that

$$\pi_{H(r)}(\Sigma \cap B^n(x, r)) \supset H(r) \cap B^n(\pi_{H(r)}(x), r\sqrt{1-\delta^2}),$$

and therefore $\mathcal{H}^m(\Sigma \cap B^n(x, r)) \geq (1-\delta^2)^{m/2} \omega(m) r^m$ for all $0 < r \leq d_s(x)$;

- (iii) the plane $W = H(d) \in G(n, m)$ is such that $\Sigma \cap A_{d,d/2}(x, \delta, W) = \emptyset$.
- (iv) Each disk $D^{n-m}(z, r; W^\perp) = z + \{v \in W^\perp : |v| \leq r\}$ with $z \in x + W$, $|z - x| \leq d\sqrt{1-\delta^2}$, and radius r such that

$$\mathbb{S}^{n-m-1}(z, r; W^\perp) := z + \{v \in W^\perp : |v| = r\} \subset A_{d,d/2}(x, \delta, W) \quad (4.5)$$

contains at least one point of Σ .

The number $d_s(x)$ is referred to as the *stopping distance*. It can be checked that the condition (4.5) for the radii of disks containing points of Σ is equivalent to

$$\frac{\delta^2}{1-\delta^2} |z-x|^2 \leq r^2 \leq d^2 - |z-x|^2 \quad \text{if} \quad \frac{d}{2} \sqrt{1-\delta^2} < |z-x| \leq d\sqrt{1-\delta^2}, \quad (4.6)$$

$$\left(\frac{d}{2}\right)^2 - |z-x|^2 \leq r^2 \leq d^2 - |z-x|^2 \quad \text{if} \quad |z-x| \leq \frac{d}{2} \sqrt{1-\delta^2}. \quad (4.7)$$

Lemma 4.4. Let $\delta(m)$ be the constant of Lemma 4.3. If $\Sigma \in \mathcal{A}(\delta)$ for some $\delta \in (0, \delta(m)]$ and $\mathcal{E}_q(\Sigma) < \infty$ for some $q > 2m$, then the numbers $d_s(x)$ satisfy

$$d(\Sigma) := \inf_{x \in \Sigma^*} d_s(x) > 0. \quad (4.8)$$

Moreover, we have

$$d(\Sigma) \geq \left(\frac{c}{\mathcal{E}_q(\Sigma)} \right)^{1/(q-2m)} =: R_1 \quad (4.9)$$

where

$$c = (2 \cdot 9^q)^{-1} \omega(m)^2 \lambda \eta^{4m+q} \quad (4.10)$$

for $\lambda = \lambda(n, m)$ and $\eta = \eta(m)$ as in Lemma 4.3.

The rest of this Section is organized as follows. We prove Lemma 4.3 in the next subsection. Then, in subsection 4.4, we derive Lemma 4.4 from Lemma 4.3, and prove Theorem 1.2.

4.3 The proof of Lemma 4.3

The proof of Lemma 4.3 is similar to the proof of Theorem 3.3 in our paper [34]. It has algorithmic nature. Proceeding iteratively, we construct an increasingly complicated set S which is centrally symmetric with respect to x and its intersection with each sphere $\partial B^n(x, r)$ is equal to the union of two or four spherical caps. The size of these caps is proportional to r but their position may change as r grows from 0 to the desired stopping distance $d_s(x)$. The interior of S contains no points of Σ but it contains numerous $(n - m - 1)$ -dimensional spheres which are nontrivially linked with Σ . Eventually, this ensures parts (ii)–(iv) of the lemma. To find a good couple (x, y) , we construct S so that $\partial S \cap (\Sigma \setminus \{x\})$ is nonempty, and one of the points in this intersection, or one of nearby points of Σ will be good enough for our purposes.

The rest of this subsection is organized as follows. We first list the conditions that have to be satisfied by η , δ , and λ . Then, we set up the plan of the whole inductive construction and describe the first step in detail. Next, we give the stopping criteria. Analyzing them, we demonstrate that when the iteration stops, then (i)–(iv) of the lemma are satisfied. If the stopping criteria do not hold, then we perform the iterative step. Due to the nature of stopping criteria the total number of steps in the iteration must be finite, since Σ is compact.

We fix a sufficiently small $\delta > 0$ (to be specified soon) and assume that Σ belongs to the class $\mathcal{A}(\delta)$ of all admissible surfaces defined in Section 2. For the sake of simplicity, we assume throughout the whole proof that $0 = x \in \Sigma^*$.

The constants. We fix the three constants $\eta = \eta(m)$, $\delta = \delta(m)$, $\lambda = \lambda(n, m)$ in $(0, \frac{1}{9})$ so that several conditions are satisfied. We first pick η and δ so small that

$$6c_2(\delta + \eta) < 6c_3(\delta + \eta) < \varepsilon_1, \quad (4.11)$$

where ε_1 , c_2 and c_3 denote the constants (depending only on m) introduced in Lemma 2.3 and Lemma 2.4 in Section 2.2.⁶ Without loss of generality we can also assume that

$$(1 - \delta^2)^{m/2} > \frac{1}{2} \quad \text{and} \quad \frac{9}{10}(1 - \delta^2)^{1/2} > \frac{2}{3}, \quad (4.12)$$

and

$$\delta \geq 5\eta. \quad (4.13)$$

Next, we let J be the minimal number such that there exist J balls

$$B_k := B_{G(n, m)}(P_k, \eta^2) = \{H \in G(n, m) : \angle(H, P_k) \leq \eta^2\}, \quad k = 1, \dots, J, \quad P_k \in G(n, m),$$

that form a covering of the whole Grassmannian $G(n, m)$. Since η depends only on m , this number J depends in fact only on n, m . Finally, once J is fixed, we let

$$\lambda = \frac{1}{3J}. \quad (4.14)$$

The construction. Proceeding iteratively, we shall construct three finite sequences:

- of compact, connected, centrally symmetric sets $S_0 \subset T_1 \subset S_1 \subset T_2 \subset S_2 \subset \dots \subset S_{N-1} \subset T_N \subset S_N \subset \mathbb{R}^n$,

⁶The stronger inequality, involving c_3 , is needed later, in applications in Section 5. Here, in this proof, just the condition $6c_2(\delta + \eta) < \varepsilon_1$ would be sufficient.

- of m -planes H_0, \dots, H_N and $H_0^*, \dots, H_{N-1}^* \in G(n, m)$ such that the angle $\angle(H_i, H_i^*) < \varepsilon_1$ for each $i = 0, \dots, N-1$, where ε_1 is the small constant of Lemma 2.3,
- and of radii $\rho_0 < \rho_1 < \dots < \rho_N$, where $\rho_N =: d_s(x)$, so ρ_N will provide the desired stopping distance for x as claimed in the statement of Lemma 4.3.

Everywhere below in this subsection, we write $V_i := H_i^\perp$ and $V_i^* := (H_i^*)^\perp$.

These sequences will be shown to satisfy the following properties:

- (A) **(Diameter of S_i grows geometrically).** We have $S_i \subset B_{\rho_i}^n \equiv B^n(0, \rho_i)$ and $\text{diam } S_i = 2\rho_i$ for $i = 0, \dots, N$. Moreover

$$\rho_i > 2\rho_{i-1} \quad \text{for } i = 1, \dots, N. \quad (4.15)$$

- (B) **(Large ‘conical caps’ in S_i and T_i).**

$$S_i \setminus B_{\rho_{i-1}} = C_{\rho_i}(\delta, H_i) \setminus B_{\rho_{i-1}} \quad \text{for } i = 1, \dots, N, \quad (4.16)$$

and

$$T_{i+1} \subset B_{\rho_i}, \quad T_{i+1} = S_i \cup \overline{A_{\rho_i, \rho_i/2}(0, \delta, H_i^*)} \quad \text{for } i = 0, \dots, N-1. \quad (4.17)$$

- (C) **(Σ does not enter the interior of S_i or T_{i+1}).**

$$\Sigma \cap \text{int } S_i = \emptyset \quad \text{for } i = 0, \dots, N, \quad (4.18)$$

$$\Sigma \cap \text{int } T_{i+1} = \emptyset \quad \text{for } i = 0, \dots, N-1. \quad (4.19)$$

Moreover, we have

$$\Sigma \cap \partial B_r \cap C(\delta, H_i^*) = \emptyset \quad \text{for } \rho_i \leq r \leq 2\rho_i, \quad i = 0, \dots, N-1. \quad (4.20)$$

- (D) **(Points of $\Sigma \setminus \{x\}$ on ∂S_i).** The intersection $\Sigma \cap \partial B_{\rho_i} \cap \partial S_i$ is nonempty for each $i = 1, \dots, N$.
- (E) **(Linking).** If $z \in H_i$ satisfies $|z| < \rho_i \sqrt{1 - \delta^2}$ and the radius $r > 0$ is chosen such that the $(n - m - 1)$ -dimensional sphere

$$\mathbb{S}^{n-m-1}(z, r; V_i) = z + \{v : v \in V_i, |v| = r\}$$

is contained in the interior of $S_i \cap (B_{\rho_i}^n \setminus B_{\rho_i/2}^n)$, then

$$\text{lk}_2(\Sigma^m, \mathbb{S}^{n-m-1}(z, r; V_i)) = 1 \quad (4.21)$$

for $i = 1, \dots, N$.

- (F) **(Big projections of $B_{\rho_i}^n \cap \Sigma$ onto H_i).** For $t \in [\rho_{i-1}, \rho_i]$, $i = 1, \dots, N$, we have

$$\pi_{H_i}(\Sigma \cap B_t^n) \supset H_i \cap B_{t\sqrt{1-\delta^2}}^n. \quad (4.22)$$

Start of the iteration. We set $S_0 := \emptyset$, $T_1 := \emptyset$, $\rho_0 := 0$ and $H_0 = H_0^* = H_1 := H_x \in G(n, m)$, where H_x stands for the *mock tangent* plane at $x = 0 \in \Sigma^*$, satisfying (H3) of Definition 2.10.

Moreover, we use the convention that our closed balls are defined as

$$B_r^n = B^n(0, r) := \overline{\{y \in \mathbb{R}^n : |y| < r\}}$$

so that the closed ball B_0 of radius zero is the empty set.

Notice that for a complete iteration start we need to define ρ_1 and S_1 in order to check Conditions (4.15) in (A), (4.16) in (B), (4.18) for $i = 1$, and (4.21)–(4.22) constituting Conditions (E) and (F). All the other conditions within the whole list are immediate for $i = 0$.

We set

$$K_t^1 := C_t(\delta, H_1). \quad (4.23)$$

With growing radii t the sets K_t^1 describe larger and larger double cones with ‘axis’ perpendicular to H_1 and fixed opening angle which is very close to π when δ is small. Now we define

$$\rho_1 := \inf\{t > \rho_0 = 0 : \Sigma \cap K_t^1 \cap \partial B_t \neq \emptyset\}, \quad (4.24)$$

and notice that since Σ^* satisfies (2.11) of condition (H3) by definition, one has $\rho_1 > r_0(x) > 0 = 2\rho_0$. This yields (4.15) in (A) for $i = 1$. Set $S_1 := K_{\rho_1}^1$; in other words we have $S_1 = C_{\rho_1}(\delta, H_1) \subset B_{\rho_1}^n$ with $\text{diam } S_1 = 2\rho_1$, so that all properties mentioned in (A) are satisfied for $i = 1$. Moreover, since we have adopted the convention that B_0 is an empty set and $\rho_0 = 0$, condition (4.16) in (B) does hold for $i = 1$. The definition of ρ_1 guarantees that there are no points of Σ in $\text{int } S_1$, implying (4.18) in (C) for $i = 1$. Condition (D) for $i = 1$ follows from the definition of ρ_1 , as Σ is a closed subset of \mathbb{R}^n .

Let us now take care of (E) and (F) for $i = 1$. To check (E), note that by Lemma 3.3 we have

$$\text{lk}_2(\Sigma^m, \mathbb{S}^{n-m-1}(0, r_1; V_1)) = 1$$

for every $r_1 > 0$, $r_1 < r_0(x) = r_0(0)$. Any sphere $\mathbb{S}^{n-m-1}(z, r; V_1)$ with z and r specified in (E) for $i = 1$ which is contained in $\text{int } S_1$ can be homotopically deformed to, say, $\mathbb{S}^{n-m-1}(0, r_1; V_1)$ with $r_1 = r_0(x)/2$; to this end, we just first move the base point z to 0 along the segment $\{tz : t \in [0, 1]\}$ in H_1 , and then adjust the radius. Notice that all $(n - m - 1)$ -spheres used to define such a homotopy are contained in $\text{int } S_1$ and therefore stay away from Σ by (4.18) in (C) for $i = 1$.

Thus, by Lemma 3.2, (E) follows for $i = 1$. (Note that in this first step we have even proved more. In fact, every sphere $\mathbb{S}^{n-m-1}(z, r; V_1)$ with $z \in H_1$, $|z| < \rho_1 \sqrt{1 - \delta^2}$ and radius r such that

$$\mathbb{S}^{n-m-1}(z, r; V_1) \subset \text{int } S_1$$

is nontrivially linked with Σ ; for $i = 1$ we do not have to restrict ourselves to spheres in $\text{int } S_1$ intersected with the annulus. This restriction, however, will be necessary at later steps.)

Invoking Lemma 3.5, we conclude that each $(n - m)$ -dimensional disk $D^{n-m}(z, r; V_1)$, with z and r as in (E) for $i = 1$, must contain at least one point of Σ . Therefore,

$$\pi_{H_1}(\Sigma \cap D^{n-m}(z, r; V_1)) = \{z\} \text{ for all } z \in H_1 \text{ with } |z| < \rho_1 \sqrt{1 - \delta^2}.$$

Since all disks $D^{n-m}(z, r; V_1)$ are contained in $B_{\rho_1}^n$ we conclude

$$H_1 \cap B_{\rho_1 \sqrt{1 - \delta^2}}^n \subset \pi_{H_1}(B_{\rho_1}^n \cap \Sigma).$$

This is the big projection property (F) for $i = 1$.

To summarize this first step, we have defined the sets $S_0 \subset T_1 \subset S_1 \subset \mathbb{R}^n$, and the planes H_0, H_0^* and H_1 which, up to now, are all identical, so that the desired estimate for the angle $\angle(H_i, H_i^*)$ holds trivially for $i = 0$. We also have defined $\rho_1 > 2\rho_0 = 0$, postponing the decision whether $N > 1$ or $N = 1$. Note that we have not defined H_1^* yet. However, (E)–(F) do hold for $i = 1$, and all those items in the list (A)–(D) for $i = 1$ which do not involve statements about T_2 or H_1^* also do hold.

We shall now discuss the stopping criteria and show how to pass to the next step of the iteration when it is necessary.

Stopping criteria and the iteration step. For the decision whether to stop the iteration or to continue it with step number $j + 1$ for $j \geq 1$, we may now assume that the sets

$$S_0 \subset T_1 \subset S_1 \subset T_2 \subset S_2 \subset \cdots \subset T_j \subset S_j \subset \mathbb{R}^n,$$

and the m -planes $H_0, \dots, H_j, H_0^*, \dots, H_{j-1}^*$ with $\angle(H_i, H_i^*) < \varepsilon_1$ for $i = 0, \dots, j - 1$, have already been defined. We also have at this point a sequence of radii $\rho_0 = 0 < \rho_1 < \cdots < \rho_j$ satisfying the growth condition (4.15) for $i = 1, \dots, j$.

The first two conditions in (A) may be assumed to hold for $i = 0, \dots, j$. In (B) we may suppose (4.16) for $i = 1, \dots, j$, in contrast to (4.17) which holds only for $i = 0, \dots, j - 1$. Similarly, we may now work with (4.19) in (C) and (4.20) in (D) for all $i = 0, \dots, j - 1$, whereas (4.18) in (D) can be assumed for $i = 0, \dots, j$. The statements in (E) and (F) can be used for $i = 1, \dots, j$.

We are going to study the geometric situations that allow us to stop the iteration right away; if this is the case, then we set $N := j$ and $d_s(x) := \rho_j = \rho_N$. Basically, there are two cases when we can stop the construction because then there is a point $y \in (B_{\rho_j} \setminus \text{int} B_{\rho_j/2}) \cap \Sigma$ such that (x, y) form a (λ, η, ρ_j) -good couple. In the third case it turns out that $\Sigma \cap B_{\rho_j} \setminus \text{int} B_{\rho_j/2}$ is contained in a thin tubular neighbourhood of some plane H_j^* , which is close to H_j and *very* close to many of the mock tangent planes H_z for points z in $B^n(x, \eta^2 \rho_j) \cap \Sigma^*$ — a priori, possibly even to all of these tangent planes. When this happens, then we set $H_{j+1} := H_j^*$, define a new radius ρ_{j+1} , new sets $T_{j+1} \subset S_{j+1}$ containing S_j , and finally check all the properties listed in (A)–(F).

The different geometric situations depend on the position of the point where the surface hits the current centrally symmetric set S_j .

Case 1. (First hit immediately gives a good couple.) This occurs if there exists at least one point $y \in \partial B_{\rho_j} \cap C(\delta, H_j) \cap \Sigma$ such that the set $S(x, y; \eta, \rho_j)$, cf. Definition 4.1 (ii), satisfies

$$\mathcal{H}^m(S(x, y; \eta, \rho_j)) \geq \lambda \omega(m) \eta^{2m} \rho_j^m. \quad (4.25)$$

If Case 1 holds, then, directly by definition, (x, y) is a (λ, η, ρ_j) -good couple. We then set $N := j$, $d_s(x) = \rho_N$, and stop the construction. It is easy to see that all conditions of Lemma 4.3 are satisfied.

If Case 1 fails, then we define the new plane H_j^* which, roughly speaking, gives a very good approximation of a significant portion of the mock tangent planes H_z for z close to x , and examine the portion of Σ contained in the closed set

$$F_j := B^n(0, 2\rho_j) \setminus \text{int} B^n(0, \rho_j/2) \quad (4.26)$$

to distinguish two more cases. In one of them the iteration can be stopped in a similar way. In the second one, the whole intersection $\Sigma \cap F_j$ might be very close to all mock tangent planes H_z so that there is no chance of finding a good couple; we have to continue the iteration then.

We begin with the definition of H_j^* . The choice of λ in (4.14) comes into play here. In one of the two remaining cases H_j^* will become the new H_{j+1} . In the other case we can stop the iteration, setting $j = N$.

Fix $y \in \partial B_{\rho_j}(x) \cap \Sigma \cap C(\delta, H_j)$. Cover the Grassmannian $G(n, m)$ by finitely many balls

$$B_k = \{H \in G(n, m) : \angle(H, P_k) \leq \eta^2\}, \quad k = 1, 2, \dots, J(n, m), \quad P_k \in G(n, m).$$

Let

$$Y_j := B_{\eta^2 \rho_j}^n \cap \Sigma^*.$$

Since we already can use the big projection property (4.22) of Condition (F) for all $i \leq j$, it follows that

$$\mathcal{H}^m(\Sigma \cap B_r^n) \geq \omega(m)(1 - \delta^2)^{m/2} r^m \quad \text{for all } r \leq \rho_j. \quad (4.27)$$

Thus, we can estimate

$$\begin{aligned} \mathcal{H}^m(Y_j) &= \mathcal{H}^m(B_{\eta^2 \rho_j}^n \cap \Sigma^*) \\ &\geq \omega(m)(1 - \delta^2)^{m/2} \eta^{2m} \rho_j^m \\ &> \frac{1}{2} \omega(m) \eta^{2m} \rho_j^m \quad \text{by (4.12).} \end{aligned}$$

Now, let

$$G_k := \{z \in Y_j : \angle(H_z, P_k) \leq \eta^2\}, \quad k = 1, 2, \dots, J.$$

Since the G_k cover Y_j , there exists at least one $k_0 \in \{1, 2, \dots, J\}$ such that

$$\begin{aligned} \mathcal{H}^m(G_{k_0}) &\geq \frac{1}{J} \mathcal{H}^m(Y_j) \\ &> \frac{1}{2J} \omega(m) \eta^{2m} \rho_j^m \\ &> \lambda \omega(m) \eta^{2m} \rho_j^m \quad \text{by (4.14).} \end{aligned} \quad (4.28)$$

We set $H_j^* := P_{k_0}$, and distinguish two more cases.

Case 2. (Some points of $\Sigma \cap F_j$ are far from H_j^* .) By this we mean that there exists a point $y \in \Sigma \cap F_j$ such that

$$|y - \pi_{H_j^*}(y)| \equiv |Q_{H_j^*}(y - x)| \geq 2\eta \rho_j. \quad (4.29)$$

If (4.29) holds, then, as in Case 1, we set $N := j$, $d_s(x) = \rho_N$, and stop the iteration. It remains to check that (x, y) is a (λ, η, ρ_j) -good couple. Condition (i) of Definition 4.1 is clearly satisfied. To check (ii) of that definition we estimate for each $z \in G_{k_0} \subset B_{\eta^2 \rho_j} \cap \Sigma^*$, using the triangle inequality,

$$\begin{aligned} |Q_{H_z}(y - z)| &= |y - z - \pi_{H_z}(y - z)| \\ &= |y - \pi_{H_j^*}(y) + \pi_{H_j^*}(y) - \pi_{H_z}(y) - z + \pi_{H_z}(z)| \\ &\geq 2\eta \rho_j - \angle(H_j^*, H_z)|y| - 2|z| \quad \text{by definition of the angle between } m\text{-planes} \\ &\geq 2\eta \rho_j - \eta^2|y| - 2\eta^2 \rho_j \quad \text{by choice of } G_{k_0} \text{ and } H_j^* \\ &> \eta \rho_j. \end{aligned}$$

(For the last inequality we just use $|y| \leq 2\rho_j$ and $\eta < 1/4$.) Therefore, $G_{k_0} \subset S(x, y, \eta, \rho_j)$. Moreover, (4.28) guarantees that $\mathcal{H}^m(G_{k_0})$ is large enough. It follows that (x, y) is a (λ, η, ρ_j) -good couple. As

before in Case 1, it is easy to see now that all conditions of Lemma 4.3 are satisfied with $H(r) = H_i$ for all $r \in (\rho_{i-1}, \rho_i]$.

If neither Case 1 nor Case 2 occurs, then we have to deal with

Case 3. (Flat position; the whole $\Sigma \cap F_j$ is very close to H_j^* .) This happens if and only if for each point $y \in \Sigma \cap F_j$ we have

$$|y - \pi_{H_j^*}(y)| \equiv |Q_{H_j^*}(y - x)| < 2\eta\rho_j. \quad (4.30)$$

Intuitively, Case 3 corresponds to the following situation: most points of $\Sigma \cap B_{\rho_j}$ are close to some fixed m -plane which is a very good approximation of H_z for many (possibly all!) points $z \in \Sigma$ close to x . We then set $H_{j+1} := H_j^*$ and have to continue the iteration.

Flat position and the passage to the next step. We shall first check that if Case 3 has occurred, then

$$\angle(V_j, V_{j+1}) \equiv \angle(H_j, H_{j+1}) \equiv \angle(H_j, H_j^*) \leq 3c_2(\delta + \eta) < \varepsilon_1. \quad (4.31)$$

In order to prove that this is indeed the case, we shall check that

$$B^n(w, 3(\delta + \eta)) \cap H_j^* \neq \emptyset \quad \text{whenever } w \in H_j \text{ and } |w| = 1. \quad (4.32)$$

Indeed, assume (4.32) were false. Fix a unit vector $w \in H_j$ such that $B^n(w, 3(\delta + \eta)) \cap H_j^*$ is empty. Let $z = sw$ for

$$s := \frac{9}{10}(1 - \delta^2)^{1/2}\rho_j \stackrel{(4.12)}{>} \frac{2}{3}\rho_j. \quad (4.33)$$

Pick

$$r := \frac{10}{9} \frac{\delta}{(1 - \delta^2)^{1/2}} |z| = \delta\rho_j < \frac{1}{9}\rho_j. \quad (4.34)$$

Then, by (4.6), the sphere $\mathbb{S}^{n-m-1}(z, r; V_j)$ is contained in the interior of the intersection of $C(\delta, H_j)$ and the annulus F_j . Thus, we may use Condition (E), (4.21) for $i = j$, and Lemma 3.5 to conclude that the disk $D^{n-m}(z, r; V_j)$ contains at least one point $y_1 \in \Sigma$. We also have $y_1 \in F_j$; this follows from the choice of z and r . Invoking (4.34) and (4.33) above, we have

$$|y_1 - z| \leq r = \delta\rho_j < 2s\delta.$$

Since $B^n(w, 3(\delta + \eta)) \cap H_j^* = \emptyset$ and $z = sw$, by scaling we have also

$$B^n(z, 3s(\delta + \eta)) \cap H_j^* = \emptyset, \quad (4.35)$$

so that the triangle inequality gives, by (4.33),

$$|y_1 - \pi_{H_j^*}(y_1)| > 3s(\delta + \eta) - 2s\delta > 3s\eta > 2\eta\rho_j.$$

This, however, is a contradiction to condition (4.30) which holds in Case 3. Hence, (4.32) holds too, and for every orthonormal basis $(e_i) \subset H_j$ the vectors $f_i := \pi_{H_{j+1}}(e_i)$ form a basis of H_{j+1} which satisfies $|e_i - f_i| \leq 3(\delta + \eta) < \varepsilon_1$. Lemma 2.3 implies that

$$\angle(H_j, H_{j+1}) \equiv \angle(H_j, H_j^*) < c_2 \cdot 3(\delta + \eta) \stackrel{(4.11)}{<} \varepsilon_1,$$

which is (4.31).

As the angle $\angle(H_j, H_{j+1}) = \angle(V_j, V_{j+1})$ is small, the cones $C(\delta, H_j)$ and $C(\delta, H_{j+1})$ have a large intersection. Indeed, for any unit vector $v \in \mathbb{R}^n$ with $|\pi_{H_{j+1}}(v)| \leq \theta$ we have $|\pi_{H_j}(v)| < \theta + 3c_2(\delta + \eta)$ by definition of $\angle(H_j, H_{j+1})$. Thus,

$$|Q_{H_j}(v)| \geq |v| - |\pi_{H_j}(v)| > 1 - \theta - 3c_2(\delta + \eta) > \delta$$

whenever $\theta < 1 - \delta - 3c_2(\delta + \eta) < 1 - \varepsilon_1$. In particular, every unit vector $v \in V_{j+1}$ belongs to the interior of $C(\delta, H_j)$.

We now define

$$T_{j+1} := S_j \cup (C_{\rho_j}(\delta, H_{j+1}) \setminus \text{int } B_{\rho_j/2}^n). \quad (4.36)$$

According to (4.30), this immediately gives the missing conditions (4.17) in (B) and (4.19) in (C) for $i = j$. To check (4.20) in (C) for $i = j$, note that in Case 3 we have

$$|Q_{H_j^*}(y)| < 2\eta\rho_j \leq 4\eta|y|$$

for each point of Σ in the annulus F_j . However, when $y \in C(\delta, H_j^*) \cap \partial B_r$ for some $\rho_j \leq r \leq 2\rho_j$, then

$$|Q_{H_j^*}(y)| \stackrel{(4.2)}{\geq} \delta|y| \stackrel{(4.13)}{\geq} 5\eta|y|,$$

so that y cannot be a point of Σ . This gives (4.20) for $i = j$.

Now the crucial thing is to define the next radius ρ_{j+1} and take care of the linking condition (4.21) for $i = j + 1$.

The next radius and homotopies from large spheres to smaller tilted ones. Set

$$K_t^{j+1} := C_t(\delta, H_{j+1}), \quad (4.37)$$

and define

$$\rho_{j+1} := \inf\{t > \rho_j : \Sigma \cap K_t^{j+1} \cap \partial B_t \neq \emptyset\}. \quad (4.38)$$

Notice that condition (4.20) guarantees that $\rho_{j+1} > 2\rho_j$. This verifies (4.15) in Condition (A) for $i = j + 1$. Now we define

$$S_{j+1} := T_{j+1} \cup (K_{\rho_{j+1}}^{j+1} \setminus \text{int } B_{\rho_j}), \quad (4.39)$$

and check that Conditions (A)–(F) are satisfied.

Indeed, $S_{j+1} \subset S_j \cup K_{\rho_{j+1}}^{j+1} \subset B_{\rho_j} \cup B_{\rho_{j+1}}$ by Condition (A) for $i = j$, which implies that (A) holds for $i = j + 1$ as well. Next,

$$S_{j+1} \setminus B_{\rho_j} = K_{\rho_{j+1}}^{j+1} \setminus B_{\rho_j} = C_{\rho_{j+1}}(\delta, H_{j+1}) \setminus B_{\rho_j},$$

since $S_j \subset B_{\rho_j}$ by Condition (A) for $i = j$. Hence (4.16) holds for $i = j + 1$. The inclusion $T_{j+1} \subset S_j \cup B_{\rho_j} \subset B_{\rho_j}$ and other conditions involving T_{j+1} have already been checked; they follow directly from the definition of T_{j+1} ; see (4.36). Using (4.18) for $i = j$ and the definition of $\rho_{j+1} > 2\rho_j$ in (4.38) we infer that (4.18) holds for $i = j + 1$, and (4.19) for $i = j$. We also have

$$\Sigma \cap \partial B_r \cap C(\delta, H_{j+1}) = \emptyset \quad \text{for each } r \in [\rho_j, \rho_{j+1}). \quad (4.40)$$

Also Condition (D) follows directly from the definition of ρ_{j+1} .

Now we turn to the proof of the linking condition, (4.21), for $i = j + 1$.

The definition (4.38) of ρ_{j+1} implies that each sphere $\mathbb{S}^{n-m-1}(z, r_0, V_{j+1})$ where $z \in H_{j+1}$, $|z| < \rho_{j+1}\sqrt{1-\delta^2}$ and the radius r_0 is such that

$$M_0 := \mathbb{S}^{n-m-1}(z, r_0, V_{j+1}) \subset \text{int} S_{j+1} \cap \{y \in \mathbb{R}^n : \rho_{j+1}/2 < |y| < \rho_{j+1}\}$$

can be homotopically deformed to

$$M_1 := \mathbb{S}^{n-m-1}(0, r_1; V_{j+1}) \subset \text{int} S_{j+1} \cap \{y \in \mathbb{R}^n : \rho_{j+1}/2 < |y| < \rho_{j+1}\}, \quad r_1^2 := |z|^2 + r_0^2,$$

without meeting any points of Σ , so that the linking invariant used in (4.21) is preserved. One of the possible homotopies is to move the base point z to 0 along the segment $z(t) = (1-t)z$, $t \in [0, 1]$, at the same time increasing the radius from $r_0 = r(0)$ to $r_1 = r(1)$ so that

$$\rho^2 := |z(t)|^2 + r(t)^2$$

remains constant for all $t \in [0, 1]$; in this way, we simply slide the $(n-m-1)$ -dimensional spheres along the surface of a fixed $(n-1)$ -sphere, staying all the time in the interior of S_{j+1} intersected with the annulus $\{y \in \mathbb{R}^n : \rho_{j+1}/2 < |y| < \rho_{j+1}\}$. By Lemma 3.2 we have

$$\text{lk}_2(\Sigma, M_0) = \text{lk}_2(\Sigma, M_1). \quad (4.41)$$

Next, we may homotopically deform the sphere M_1 to another sphere of radius r_2 ,

$$M_2 := \mathbb{S}^{n-m-1}(0, r_2; V_{j+1}), \quad r_2 = \frac{8}{9}\rho_j \in (\rho_j/2, \rho_j).$$

We just shrink the radius linearly, staying all the time in the $(n-m)$ -dimensional subspace V_{j+1} . It is clear that all the flat spheres realizing this homotopy $M_1 \sim M_2$ stay in the interior of S_{j+1} (by (4.18) for $i = j+1$ and the definition of T_{j+1} in (4.36)) and do not contain any points of Σ , so that, again by Lemma 3.2,

$$\text{lk}_2(\Sigma, M_1) = \text{lk}_2(\Sigma, M_2). \quad (4.42)$$

But M_2 can be homotoped — still in the interior of S_{j+1} — to another sphere,

$$M_3 := \mathbb{S}^{n-m-1}(0, r_2; V_j),$$

which has the same radius r_2 but is slightly tilted; therefore,

$$\text{lk}_2(\Sigma, M_2) = \text{lk}_2(\Sigma, M_3). \quad (4.43)$$

To check this, we perform two steps. First we move each point y of $M_2 \subset V_{j+1}$ along the segment that joins y to its projection $\pi_{V_j}(y)$. This gives an ellipsoid which is nearly spherical and has all axes at least $(1-\varepsilon_1)r_2$ because of the condition (4.31) for the angle between V_j and V_{j+1} . Next, we continuously blow up this ellipsoid, moving each of its points along the rays that emanate from 0 to points of M_3 . Because of the smallness condition (4.11) for the constants that we use, each segment I_y with one endpoint at $y \in M_2$, $|y| = \frac{8}{9}\rho_j$, and the other at $\pi_{V_j}(y)$ is certainly contained in the interior of S_j (i.e. far away from Σ), as

$$|y - \pi_{V_j}(y)| \leq \angle(V_j, V_{j+1})|y| \leq 3c_2(\delta + \eta)r_2 < 3c_2(\delta + \eta)\rho_j < \frac{1}{3}\rho_j,$$

so that $|\pi_{V_j}(y)| > \frac{8}{9}\rho_j - \frac{1}{3}\rho_j > \frac{1}{2}\rho_j > 2\rho_{j-1}$. Thus, invoking Lemma 3.2 one more time, and applying the inductive assumption, i.e. the linking condition (4.21) for $i = j$, we finally obtain

$$\text{lk}_2(\Sigma, M_0) = \text{lk}_2(\Sigma, M_3) = 1$$

This gives (4.21) of (E) for $i = j + 1$.

It is now easy to establish the big projection property of (F) for $i = j + 1$. We do this as in the first step of the proof: invoking Lemma 3.5, we conclude that each flat $(n - m)$ -dimensional disk $D^{n-m}(z, r; V_{j+1})$, with z and r as in (E) for $i = j + 1$, must contain at least one point of Σ . Therefore,

$$\pi_{H_{j+1}}(\Sigma \cap D^{n-m}(z, r; V_{j+1})) = \{z\} \text{ for all } z \in H_{j+1} \text{ with } |z| < \rho_{j+1} \sqrt{1 - \delta^2}.$$

All disks $D^{n-m}(z, r; V_{j+1})$ are contained in $B_{\rho_{j+1}}^n$ so that

$$H_{j+1} \cap B_{\rho_{j+1} \sqrt{1 - \delta^2}}^n \subset \pi_{H_{j+1}}(B_{\rho_{j+1}}^n \cap \Sigma).$$

This gives (4.22) in (F) for $i = j + 1$, and finishes the proof of all conditions in the list (A)–(F) in the iteration step.

Since we have established Condition (E) in the iteration step and (4.15) holds, too, we can deduce that Case 3 can happen only finitely many times, depending on the position x on Σ and on the shape and size of Σ :

$$\text{diam} \Sigma \geq \rho_i > 2\rho_{i-1} > \cdots > 2^{i-1}\rho_1 > 2^{i-1}r_0(x),$$

whence the maximal number of iteration steps is bounded by

$$1 + \log(\text{diam} \Sigma / r_0(x)) / \log 2.$$

This concludes the consideration of Case 3, and the whole proof of Lemma 4.3 . \square

4.4 Bounds for the stopping distances and uniform Ahlfors regularity

We shall now derive Lemma 4.4 and Theorem 1.2 from Lemma 4.3. This is a relatively easy task at this stage. We shall just rely on estimates for the \mathcal{E}_q -energy in the neighbourhood of a good couple $(x, y) \in \Sigma \times \Sigma$.

Proof of Lemma 4.4. Since $\mathcal{A}(\delta) \subset \mathcal{A}(\delta')$ for $\delta \leq \delta'$, we assume from now on that $\delta = \delta(m)$ is the constant of Lemma 4.3.

Fix $\varepsilon > 0$ small (to be specified later on). Assume that $d(\Sigma) = \inf_{\Sigma^*} d_s < \varepsilon$ and select a point $x \in \Sigma^*$ such that $d_s(x) < \varepsilon$. Use Lemma 4.3 to select a (λ, η, d) -good couple $(x, y) \in \Sigma \times \Sigma$. Let

$$S := S(x, y; \eta, d_s(x))$$

be as in Definition 4.1 (ii), and let $B := B(y, \eta^2 d_s(x))$. Applying Lemma 4.2 we estimate

$$\begin{aligned} \mathcal{E}_q(\Sigma) &\geq \int_S \int_{\Sigma \cap B} \left(\frac{1}{R_{\text{tp}}(z, w)} \right)^q d\mathcal{H}^m(w) d\mathcal{H}^m(z) \\ &> \mathcal{H}^m(S) \mathcal{H}^m(\Sigma \cap B) \left(\frac{\eta}{9d_s(x)} \right)^q \quad \text{by Lemma 4.2} \\ &\geq \lambda \omega(m) \eta^{2m} d_s(x)^m \cdot K_\Sigma \eta^{2m} d_s(x)^m \left(\frac{\eta}{9d_s(x)} \right)^q \quad \text{by Definitions 4.1 and 2.10} \\ &= K_\Sigma 9^{-q} \lambda \eta^{4m+q} d_s(x)^{2m-q}, \end{aligned}$$

which implies

$$\begin{aligned}\varepsilon^{q-2m} &> d_s(x)^{q-2m} \\ &> K_\Sigma \lambda \eta^{4m+q} 9^{-q} \mathcal{E}_q(\Sigma)^{-1},\end{aligned}$$

a contradiction for

$$\varepsilon := \left(\frac{1}{2} K_\Sigma \lambda \eta^{4m+q} 9^{-q} \mathcal{E}_q(\Sigma)^{-1} \right)^{1/(q-2m)}.$$

This proves the first part of the lemma.

Now, for an arbitrarily small $\sigma \in (0, 1)$ pick $x_0 \in \Sigma^*$ such that $d(\Sigma) \leq d_0 = d_s(x_0) < (1 + \sigma)d(\Sigma)$. Select $y_0 \in \Sigma$ so that (x_0, y_0) is a (λ, η, d_0) -good couple. We have $d_s(y_0) \geq d(\Sigma) > d_0/(1 + \sigma)$, so that by Lemma 4.3 (ii)

$$\mathcal{H}^m(\Sigma \cap B^n(y, r)) \geq (1 - \delta^2)^{m/2} \omega(m) r^m \geq \frac{1}{2} \omega(m) r^m$$

certainly holds for $r = \eta^2 d_0 < d_0/(1 + \sigma)$ since $\eta \ll 1$ by (4.11). Estimating the energy one more time, as before, we obtain

$$\begin{aligned}\mathcal{E}_q(\Sigma) &\geq \int_{S(x_0, y_0; \eta, d_0)} \int_{\Sigma \cap B(y_0, \eta^2 d_0)} \left(\frac{1}{R_{\text{tp}}(z, w)} \right)^q d\mathcal{H}^m(w) d\mathcal{H}^m(z) \\ &> \frac{\lambda}{2 \cdot 9^q} \omega(m)^2 \eta^{4m+q} d_0^{2m-q} \quad \text{by Lemma 4.2.}\end{aligned}$$

Thus,

$$(1 + \sigma)^{q-2m} d(\Sigma)^{q-2m} > d_0^{q-2m} > c \mathcal{E}_q(\Sigma)^{-1},$$

where $c = (2 \cdot 9^q)^{-1} \omega(m)^2 \lambda \eta^{4m+q}$, as in (4.10). Letting $\sigma \rightarrow 0$, we obtain (4.9) and conclude the whole proof. \square

Proof of Theorem 1.2. By the lower bound (4.9) for stopping distances, the inequality

$$\mathcal{H}^m(\Sigma \cap B(x, r)) \geq (1 - \delta^2)^{m/2} \omega(m) r^m \geq \frac{1}{2} \omega(m) r^m$$

holds for each $x \in \Sigma^*$ and each $r \leq d(\Sigma) \leq d_s(x)$. By density of Σ^* in Σ , we obtain

$$\mathcal{H}^m(\Sigma \cap B(x, r)) \geq \frac{1}{2} \omega(m) r^m$$

for all $x \in \Sigma$ and $r \leq d(\Sigma)$. This implies Theorem 1.2 with

$$a_1 := \left(\frac{\lambda \omega(m)^2 \eta^{4m+q}}{2 \cdot 9^q} \right)^{1/(q-2m)},$$

where $\lambda = \lambda(n, m)$ and $\eta = \eta(m)$ are the constants introduced in Lemma 4.3.

Remark 4.5. As we have already mentioned in the introduction, the proof above yields a result which is stronger than the formal statement of Theorem 1.2. In fact, the result holds also for all $\Sigma \in \mathcal{A}(\delta)$ with $0 < \delta \leq \delta(m)$, where $\delta(m)$ is the positive constant of Lemma 4.3, and this is a wider class of sets than the one we used in the introduction.

5 Existence of tangent planes

In this section we prove that for each point $x \in \Sigma$ there exists a plane $T_x \Sigma \in G(n, m)$ such that $\text{dist}(x', x + T_x \Sigma) = o(|x' - x|)$ for $x' \in \Sigma$, $x' \rightarrow x$. Moreover, the mapping $x \mapsto T_x \Sigma$ is of class C^κ , $\kappa = (q - 2m)/(q + 4m) > 0$. A posteriori it turns out that if $\delta > 0$ is small enough and $\Sigma \in \mathcal{A}(\delta)$ is an admissible surface with $\mathcal{E}_q(\Sigma) < \infty$ for some $q > 2m$, then the mock tangent planes H_x defined a.e. on Σ must coincide with the classically understood $T_x \Sigma$.

The idea is to combine the results of the previous section with energy bounds and show that the P. Jones' β -numbers of Σ satisfy a decay estimate of the form $\beta_\Sigma(x, r) \lesssim E^{1/(q+4m)} r^\kappa$. This alone would not be enough, but we already know that Σ has big projections. Adding this ingredient, we are able to prove that Σ is in fact a $C^{1,\kappa}$ -manifold. Moreover, in each ball of radius $\approx \mathcal{E}_q(\Sigma)^{-1/(q-2m)}$ centered at $x \in \Sigma$ the surface Σ is a graph of a $C^{1,\kappa}$ function $f: P \rightarrow P^\perp$ over $P = T_x \Sigma \in G(n, m)$.

The core of this section is formed by an iterative construction, presented in Section 5.3, which yields the existence of tangent planes and estimates for their oscillation. At each iteration step, we need to check that the β -numbers decrease sufficiently fast as the length scale shrinks to zero. At the same time, we have to guarantee that the linking conditions which imply the existence of big projections are also satisfied. To make the presentation of that proof easier to digest, we introduce an ad-hoc notion of *trapping boxes* (Section 5.1) and prove an auxiliary lemma which is then used in the iteration.

5.1 Trapping boxes

Everywhere in this section R_1 denotes the radius specified in Theorem 1.2 ascertaining the uniform Ahlfors regularity of surfaces with bounded energy.

For the rest of the whole section, we fix $\delta, \eta > 0$ small so that (4.11) is satisfied and all claims of Lemma 4.3 are fulfilled.

Definition 5.1. Assume that $\Sigma \in \mathcal{A}(\delta)$, $x \in \Sigma$, $0 < r < R_1$, $\theta \in (0, \delta]$ and $H \in G(n, m)$. We say that a closed set $F \subset B^n(x, r)$ is a (θ, H) -trapping box for Σ in $B^n(x, r)$ if and only if the following conditions are satisfied:

- (i) $\Sigma \cap B^n(x, r) \subset F$;
- (ii) $\{y \in B^n(x, r) : \text{dist}(y, x + H) \leq \theta r\} \subset F$;
- (iii) if $z \in x + H$ satisfies $|z - x| < (1 - \theta^2)^{1/2} r$, then there exists a $t > 0$ such that $t^2 + |z - x|^2 < r^2$, the sphere $\mathbb{S}^{n-m-1}(z, t; H^\perp)$ is contained in the interior of $B^n(x, r) \setminus F$ and

$$\text{lk}_2(\Sigma, \mathbb{S}^{n-m-1}(z, t; H^\perp)) = 1.$$

Thus, informally, a trapping box is a subset of $B = B^n(x, r)$ which is at least as large as a cylindrical neighbourhood of $x + H$ in B (of size specified by the parameter θ), and gives us some control of the location of $\Sigma \cap B$ and of its projections onto H .

If $\Sigma \in \mathcal{A}(\delta)$, $x \in \Sigma^*$ and $H = H_x$ is given by Condition (H3) of Definition 2.10, then — for radii $r < r_0(x)$ — a simple example of a trapping box is provided by the cylinder

$$\{y \in B^n(x, r) : \text{dist}(y, x + H) \leq \delta r\}.$$

It satisfies all conditions of Definition 5.1 for $\theta = \delta$; in particular, Lemma 3.3 guarantees Condition (iii).

Another example is given by the following.

Proposition 5.2. *Let $\delta(m)$ be the small constant of Lemma 4.3. Assume that $\Sigma \in \mathcal{A}(\delta)$, $\delta \in (0, \delta(m)]$, $\mathcal{E}_q(\Sigma) \leq E$, and R_1 denotes the radius specified in Theorem 1.2. Then, for each $x \in \Sigma$ and each $r \in (0, R_1)$ there exists a plane $H \in G(n, m)$ such that*

$$F := \{y \in B^n(x, r) : \text{dist}(y, x + H) \leq \delta r\} \cup B^n(x, r/2) \quad (5.1)$$

is a (δ, H) -trapping box for Σ in $B^n(x, r)$.

Proof. One can check that conditions (A)–(F) stated at the beginning of the proof of Lemma 4.3 combined with the lower bound for stopping distances obtained in Lemma 4.4 imply the statement of Proposition 5.2 for all points $x \in \Sigma^*$. (To see this, look at condition (4.30) of Case 3, which is the only case when the iterative construction is continued. It has been designed in such a way that the union of $\{y \in B^n(x, r) : \text{dist}(y, x + H_j^*) \leq 2\eta\rho_j\}$ and $B^n(x, \rho_j/2)$ be a trapping box for Σ in $B(x, 2\rho_j)$; condition (E), cf. (4.21), implies the existence of *many* spheres linked with Σ so that (iii) of Definition 5.1 is also satisfied. Since $\eta \leq \delta/5$ by (4.13), the claim of the proposition holds for all $r \in [\rho_j, \rho_{j+1}]$ with $H = H_j^*$, and we can certainly increase r up to the infimum $d(\Sigma)$ of all stopping distances, which satisfies $d(\Sigma) \geq R_1$ by Lemma 4.4.)

Assume now that $x \notin \Sigma^*$. Fix $r \in (0, R_1)$ and select a sequence $x_l \in \Sigma^*$, $x_l \rightarrow x$ as $l \rightarrow \infty$. For each $l = 1, 2, \dots$, let H_l whose existence is given by the statement of the proposition at points $x_l \in \Sigma^*$. Passing to a further subsequence, we may assume that $H_l \rightarrow H \in G(n, m)$ as $l \rightarrow \infty$. The trapping boxes F_l corresponding to x_l and H_l via (5.1) converge then in Hausdorff distance to a closed set F given by (5.1) for x and H . Since Σ is closed, $\Sigma \cap B^n(x, r)$ must be contained in F . Condition (ii) of Definition 5.1 is trivially satisfied, and condition (iii) is easily verified by using homotopical invariance of the linking number as we already did before (one has to slightly tilt the spheres in $B^n(x, r) \setminus F$ to obtain spheres in $B^n(x_l, r) \setminus F_l$). \square

The main idea of this section is to show that once we have a trapping box of the form (5.1), possibly with δ replaced by some smaller number $\theta > 0$, then, under a certain balance condition for φ , r and the energy of Σ , we can perturb the plane H slightly to a new position H_1 and find a smaller, cylindrical (φ, H_1) -trapping box. We make this precise in the next subsection.

5.2 Energy bounds and trapping boxes in small scales

We introduce two new constants

$$c_4 := 3(c_3 + 1), \quad c_5 := \frac{16m \cdot 9^q}{\omega(m)^2}. \quad (5.2)$$

Recall from Lemma 2.4 that the constant $c_3 = 14m \cdot 20^m$ depends on m only.

Lemma 5.3. *Assume that $H \in G(n, m)$, $x \in \Sigma$, $0 < r < R_1$, $0 < \theta \leq \delta$, $q > 2m$. Let $\Sigma \in \mathcal{A}(\delta)$, $\delta \in (0, \delta(m)]$ be an admissible surface with $\mathcal{E}_q(\Sigma) \leq E$. Suppose that*

$$F_{\theta, r}(H) := \{y \in B^n(x, r) : \text{dist}(y, x + H) \leq \theta r\} \cup B^n(x, r/2)$$

is a (θ, H) -trapping box for Σ in $B^n(x, r)$. If $0 < \varphi < 1/(6c_4)$ satisfies the balance condition

$$\varphi^{4m+q} r^{2m-q} \geq c_5 E, \quad (5.3)$$

then there exists a plane $H_1 \in G(n, m)$ such that

(i) $\angle(H, H_1) \leq 2c_2\theta$;

(ii) *The cylinder*

$$F := \{y \in B_{2r}^n : \text{dist}(y, x + H_1) \leq c_4\varphi \cdot 2r\} \quad (5.4)$$

is a $(c_4\varphi, H_1)$ -trapping box for Σ in $B^n(x, 2r)$.

The main point is that once we fix a finite energy level E , and r sufficiently small, then the condition $q > 2m$ guarantees that there are numbers $\varphi > 0$ which satisfy the balance condition (5.3) and are such that $c_4\varphi$ is (much) smaller than θ . Since the angle $\angle(H, H_1)$ is controlled due to (i), the lemma can be applied iteratively. This will be done in the next subsection.

Remark 5.4. *If we fix an arbitrary point $y \in (B^n(x, r) \cap \Sigma) \subset F_{\theta, r}(H)$ such that $\frac{9}{10}(1 - \theta^2)^{1/2}r \leq |y - x| < r$, then the plane H_1 in Lemma 5.3 can be chosen so that $y - x \in H_1$, as can be seen from the first step of the following proof.*

Proof of Lemma 5.3. Fix an arbitrary orthonormal basis (e_1, \dots, e_m) of H and let

$$d := \frac{9}{10}(1 - \theta^2)^{1/2}r.$$

Since $\theta \leq \delta$, we have $d > \frac{2}{3}r$ by (4.12). Set $z_i = de_i$, $i = 1, \dots, m$.

Step 1. Choice of H_1 . Using Condition (iii) of Definition 5.1, Lemma 3.2 and Lemma 3.5, we conclude that each disk

$$D_i := D^{n-m}(z_i, \theta r; H^\perp), \quad i = 1, \dots, m,$$

contains⁷ a point $y_i \in \Sigma$. Set $H_1 = \text{span}(y_1, \dots, y_m)$. Letting $h_i = d^{-1}y_i$, we use $\theta \leq \delta$ and (4.11) to estimate

$$|h_i - e_i| = d^{-1}|y_i - z_i| \leq \frac{\theta r}{d} < 2\theta < \frac{\varepsilon_1}{2},$$

and invoke Lemma 2.3 to obtain $\angle(H, H_1) < 2c_2\theta$. (This initial step of the proof shows why Remark 5.4 is satisfied. We can work with an orthonormal basis e_i such that $e_1 = \pi_H(y)/|\pi_H(y)|$.)

Now, set $\Lambda = 1/4m$.

Step 2. For z near 0, most of the H_z are close to H_1 . We shall establish the following: for each $i = 1, \dots, m$, the couple of points $x = 0$ and y_i is not a (Λ, φ, r) -good couple.

Assume that the opposite were true and for some $i = 1, \dots, m$ we had a (Λ, φ, r) -good couple (x, y_i) . Then, using the two estimates

$$\mathcal{H}^m(S(0, y_i; \varphi, r)) \geq \Lambda \omega(m) \varphi^{2m} r^m, \quad (5.5)$$

$$\mathcal{H}^m(\Sigma \cap B^n(y_i, \varphi^2 r)) \geq \frac{1}{2} \omega(m) \varphi^{2m} r^m, \quad (5.6)$$

where (5.6) comes from Theorem 1.2, and the inequality of Lemma 4.2 to estimate $1/R_{\text{tp}}$, we would obtain a lower bound for the energy,

$$\begin{aligned} E &\geq \int_{S(0, y_i; \varphi, r)} \int_{\Sigma \cap B^n(y_i, \varphi^2 r)} \frac{1}{R_{\text{tp}}^q(z, w)} d\mathcal{H}^m(w) \otimes d\mathcal{H}^m(z) \\ &\geq \frac{\Lambda}{2} \omega(m)^2 \varphi^{4m} r^{2m} \left(\frac{1}{9} \frac{\varphi}{r} \right)^q \\ &= \frac{\omega(m)^2}{8m \cdot 9^q} \varphi^{4m+q} r^{2m-q} \geq 2E \end{aligned} \quad (5.7)$$

⁷There are points of Σ in all disks with slightly larger radii, and Σ is closed.

by (5.2) and the balance condition (5.3); this contradiction proves that the claim of Step 2 does hold. In particular, since the condition $r/2 < |y_i| < 2r$ is satisfied for each i , we have

$$\begin{aligned} \mathcal{H}^m\left(\bigcup_{i=1}^m S(0, y_i; \varphi, r)\right) &\leq \sum_{i=1}^m \mathcal{H}^m(S(0, y_i; \varphi, r)) \\ &< m\Lambda\omega(m)\varphi^{2m}r^m = \frac{1}{4}\omega(m)\varphi^{2m}r^m. \end{aligned} \quad (5.8)$$

Step 3. The new box contains $\Sigma \cap B^n(x, 2r)$. We shall show that the cylinder F defined by (5.4) contains $\Sigma \cap B_{2r}^n$.

Again, we argue by contradiction. Suppose that there exists $\zeta \in \Sigma \cap B_{2r}^n$ such that $\zeta \notin F$. Set

$$G := (\Sigma^* \cap B^n(0, \varphi^2 r)) \setminus \bigcup_{i=1}^m S(0, y_i; \varphi, r).$$

By Theorem 1.2 and (5.8), we have

$$\mathcal{H}^m(G) \geq \frac{1}{4}\omega(m)\varphi^{2m}r^m, \quad (5.9)$$

and due to the definition of $S(0, y_i; \varphi, r)$ we know that

$$|Q_{H_z}(y_i - z)| < \varphi r, \quad z \in G, \quad i = 1, \dots, m. \quad (5.10)$$

Fix $z \in G$. (5.10) yields $|Q_{H_z}(y_i)| < \varphi r + |z| \leq 2\varphi r$. Thus, the basis v_1, \dots, v_m of $W := H_z$ given by

$$v_i = y_i - Q_{H_z}(y_i), \quad i = 1, \dots, m,$$

satisfies $|v_i - y_i| \leq 2\varphi r$ for each i . Letting $w_i := d^{-1}v_i$, we check that

$$|w_i - h_i| = d^{-1}|v_i - y_i| \leq \frac{2\varphi r}{d} < 3\varphi \ll \frac{\varepsilon_1}{2},$$

as $6\varphi < (c_4)^{-1} \ll 10^{-1}(1 + 10^m)^{-1} = \varepsilon_1$. Invoking Lemma 2.4 for $H = H_1$ and $W = H_z$, we conclude that

$$\angle(H_1, W) \equiv \angle(H_1, H_z) \leq 3c_3\varphi.$$

Now, since $\zeta \notin F$, we have $|Q_{H_1}(\zeta)| > 2c_4\varphi r$, and

$$|Q_{H_1}(\zeta) - Q_W(\zeta)| \leq \angle(H_1, W)|\zeta| \leq 6c_3\varphi r.$$

Thus, for $w \in B^n(\zeta, \varphi^2 r)$ and $z \in G \subset B(0, \varphi^2 r)$

$$\begin{aligned} |Q_{H_z}(w - z)| &\equiv |Q_W(w - z)| = |Q_W(\zeta - z) - Q_W(\zeta - w)| \\ &\geq |Q_W(\zeta)| - |z| - \varphi^2 r \\ &\geq |Q_{H_1}(\zeta)| - |Q_{H_1}(\zeta) - Q_W(\zeta)| - 2\varphi^2 r \\ &> 2c_4\varphi r - 6c_3\varphi r - 2\varphi^2 r \geq 5\varphi r, \end{aligned}$$

since c_4 satisfies (5.2) and $2\varphi^2 \leq \varphi$. On the other hand, we certainly have $|w - z| \leq 3r$ for every point $w \in B^n(\zeta, \varphi^2 r)$. This yields

$$\frac{1}{R_{\text{tp}}(z, w)} = \frac{2|Q_{H_z}(w - z)|}{|w - z|^2} > \frac{2 \cdot 5\varphi r}{(3r)^2} > \frac{\varphi}{r}, \quad \text{for } z \in G, w \in B^n(\zeta, \varphi^2 r).$$

We may now estimate the energy analogously to (5.7) and obtain would obtain a lower bound for the energy,

$$\begin{aligned}
E &\geq \int_G \int_{\Sigma \cap B^n(\zeta, \varphi^2 r)} \frac{1}{R_{\text{tp}}^q}(z, w) d\mathcal{H}^m(w) d\mathcal{H}^m(z) \\
&> \frac{1}{4 \cdot 2} \omega(m)^2 \varphi^{4m} r^{2m} \left(\frac{\varphi}{r}\right)^q \\
&> \frac{2}{c_5} \varphi^{4m+q} r^{2m-q} \geq 2E.
\end{aligned} \tag{5.11}$$

This is again a contradiction, proving that $\Sigma \cap B_{2r}^n \subset F$.

Step 4. The linking condition. Since we have established $\angle(H, H_1) \leq 2c_2\theta \leq 2c_2\delta^{(4.11)} < \varepsilon_1$ in the first step of the proof, the sphere

$$M_1 := \mathbb{S}^{n-m-1}(0, \frac{8}{9}r; H_1^\perp)$$

is contained in the interior of $B_r^n \setminus F$ and we have $\text{dist}(M_1, \Sigma) \geq \frac{8}{9}r - 2c_4\varphi r > \frac{5}{9}r$, since all points of $\Sigma \cap B_{2r}^n$ are in the cylinder F defined in (5.4), and $\varphi < 1/6c_4$. Thus, we may deform M_1 homotopically to

$$M_0 := \mathbb{S}^{n-m-1}(0, \frac{8}{9}r; H^\perp),$$

so that the whole family of spheres realizing the homotopy stays in $B_r^n \setminus F$, i.e. far away from Σ . (This can be done precisely as in the verification of (4.43) at the end of the proof of Lemma 4.3: we move the points of M_1 to their projections onto H^\perp , and then deform the resulting ellipsoid to obtain the round sphere M_0 .)

Thus,

$$\text{lk}_2(M_1, \Sigma) = 1$$

by Lemma 3.2. Now, every other sphere $\mathbb{S}^{n-m-1}(z, t; H_1^\perp)$, with $z \in H_1$, $|z| < (1 - (c_4\varphi)^2)^{1/2} \cdot 2r$ and $c_4\varphi \cdot 2r < t < (2r)^2 - |z|^2$, i.e. every $(n-m-1)$ -sphere parallel to H_1^\perp and contained in the interior of $B_{2r}^n \setminus F$, can obviously be deformed homotopically to M_1 without hitting points of Σ , since $\Sigma \cap B^n \subset F$. Thus, again by Lemma 3.2, we conclude that Condition (iii) of Definition 5.1 is satisfied for F in B_{2r}^n .

This completes the whole proof of the lemma. \square

5.3 The tangent planes arise: an iterative construction

In this subsection, we apply Lemma 5.3 iteratively and prove the following.

Theorem 5.5. *Let $\delta(m)$ be the constant of Lemma 4.3. Assume that $\Sigma \in \mathcal{A}(\delta)$ for some $\delta \in (0, \delta(m)]$, $\mathcal{E}_q(\Sigma) \leq E$, $q > 2m$. Then Σ is an embedded m -dimensional submanifold of class $C^{1,\kappa}$, $\kappa = (q - 2m)/(q + 4m)$.*

In fact, Theorem 5.5 will be just a corollary of another result, which gives a lot of more precise, quantitative information.

Theorem 5.6. *Let $\delta(m)$ be the constant of Lemma 4.3. Assume that $\Sigma \in \mathcal{A}(\delta)$ for some $\delta \in (0, \delta(m)]$, $\mathcal{E}_q(\Sigma) \leq E$, $q > 2m$. Then for each $x \in \Sigma$ there exists a unique plane $T_x\Sigma \in G(n, m)$ (which we refer to as tangent plane of Σ at x) such that*

$$\text{dist}(x', x + T_x\Sigma) \leq C(n, m, q, E)|x' - x|^{1+\kappa} \quad \text{for all } x' \in \Sigma, \quad x' \rightarrow x, \tag{5.12}$$

Moreover, there exists a constant $a_2 = a_2(n, m, q) > 0$ with the following property.

Whenever $x, y \in \Sigma$ are such that

$$0 < d := |x - y| < R_2 := a_2 E^{-1/(q-2m)}, \quad (5.13)$$

then

$$\angle(T_x \Sigma, T_y \Sigma) < c_6 E^{1/(q+4m)} |x - y|^\kappa, \quad \kappa = \frac{q-2m}{q+4m} \quad (5.14)$$

for some constant c_6 depending only on n, m and q . Moreover, $U := \Sigma \cap \text{int} B^n(x, R_2)$ is an open m -dimensional topological disk, the orthogonal projection $\pi_{T_x \Sigma}$ onto $T_x \Sigma$ restricted to U is injective, and each cylinder

$$K_N := \{w \in B^n(x, 2d_N) : \text{dist}(w, x + T_x \Sigma) \leq \beta_N \cdot 2d_N\}, \quad N = 1, 2, \dots \quad (5.15)$$

with

$$d_N := \frac{d}{5^{N-1}}, \quad \beta_N = c_6 E^{1/(q+4m)} d_N^\kappa < \frac{1}{20} \quad (5.16)$$

is a $(\beta_N, T_x \Sigma)$ -trapping box for Σ in $B^n(x, 2d_N)$.

Proof. A rough plan of the proof is the following. We shall first show, using Lemma 5.3 iteratively, that for each $x \in \Sigma$ there exists a plane $H_x^* \in G(n, m)$ such that for x, y sufficiently close we have $\angle(H_x^*, H_y^*) \lesssim |x - y|^\kappa$. As a byproduct, we shall obtain a sequence of trapping boxes around each H_x^* , allowing us to show that H_x^* is in fact unique. Finally, we set $T_x \Sigma = H_x^*$ and verify the statements concerning $\pi_{T_x \Sigma}$.

Step 1. Fix $x, y \in \Sigma$ and assume that (5.13) does hold for a sufficiently small positive constant a_2 that shall be specified later on. Fix $r_1 > 0$ such that

$$\frac{2}{3} r_1 < \frac{9}{10} (1 - \delta^2)^{1/2} r_1 \leq |x - y| = d < r_1 < R_2. \quad (5.17)$$

Invoking Proposition 5.2 for x and $r = r_1$, we obtain a plane $H \in G(n, m)$ such that

$$F := \{w \in B^n(x, r_1) : \text{dist}(w, x + H) \leq \delta r_1\} \cup B^n(x, r_1/2)$$

is a (δ, H) -trapping box for Σ in $B^n(x, r_1)$.

Now, for $N = 1, 2, \dots$ we set

$$r_N := \frac{r_1}{5^{N-1}}, \quad (5.18)$$

$$\varphi_N := c_5^{1/(q+4m)} E^{1/(q+4m)} r_N^\kappa, \quad \kappa = \frac{q-2m}{q+4m}, \quad (5.19)$$

$$\theta_N := 10c_4 \varphi_N. \quad (5.20)$$

We have $\varphi_N \lesssim r_N^\kappa \rightarrow 0$ as $N \rightarrow \infty$; the constant a_2 will be chosen later, in (5.31) below, so small that δ and r_1 shall satisfy the assumptions of Lemma 5.3. The choice of r_1 guarantees that

$$r_N^\kappa \leq \left(\frac{3}{2}\right)^\kappa d_N^\kappa < \frac{3}{2} d_N^\kappa \quad \text{for all } N = 1, 2, \dots \quad (5.21)$$

Apply Lemma 5.3 and Remark 5.4 with $\theta = \delta$, $r = r_1$ and $\varphi = \varphi_1$ to choose $H_1 \in G(n, m)$ such that $y - x$ in H_1 and the cylinder

$$F_1 := \{w \in B^n(x, 2r_1) : \text{dist}(w, x + H_1) \leq 2c_4 \varphi_1 r_1\}$$

is a $(c_4\varphi_1, H_1)$ -trapping box for Σ in $B^n(x, 2r_1)$. (The plane H_1 will serve, roughly speaking, as a sort of average position for all tangent planes to Σ in $B^n(x, r_1)$.)

Step 2. The choice of H_x^* . Since $r_2 = r_1/5$, we have $2c_4\varphi_1 r_1 = \theta_1 r_2$, and the intersection $F_1 \cap B^n(x, r_2)$ provides a (θ_1, H_1) -trapping box for Σ in $B^n(x, r_2)$. Invoking Lemma 5.3 again, we find a plane $H_2 \in G(n, m)$ such that

$$\angle(H_2, H_1) \leq 2c_2\theta_1$$

and the cylinder $F_2 := \{w \in B^n(x, 2r_2) : \text{dist}(w, x + H_2) \leq 2c_4\varphi_2 r_2\}$ is a $(c_4\varphi_2, H_2)$ -trapping box for Σ in $B^n(x, 2r_2)$. Proceeding inductively, we find a sequence of planes $H_N \in G(n, m)$ such that for each $N = 1, 2, \dots$ the cylinder

$$F_N := \{w \in B^n(x, 2r_N) : \text{dist}(w, x + H_N) \leq 2c_4\varphi_N r_N\} \quad (5.22)$$

is a $(c_4\varphi_N, H_N)$ -trapping box for Σ in $B^n(x, 2r_N)$ and we have the estimate

$$\angle(H_{N+1}, H_N) \leq 2c_2\theta_N \quad \text{for all } N = 1, 2, \dots \quad (5.23)$$

Since $\sum \theta_N < \infty$, the planes H_N converge to some plane $H_x^* \in G(n, m)$ such that

$$\begin{aligned} \angle(H_x^*, H_N) &\leq \sum_{j=N}^{\infty} \angle(H_{j+1}, H_j) \\ &\leq 20c_2c_4 \sum_{j=N}^{\infty} \varphi_j \quad \text{by (5.23) and (5.20)} \\ &= 20c_2c_4c_5^{1/(q+4m)} E^{1/(q+4m)} r_N^\kappa \sum_{i=0}^{\infty} 5^{-i\kappa} \quad \text{by (5.18) and (5.19)} \\ &\leq Ar_N^\kappa, \quad N = 1, 2, \dots, \end{aligned} \quad (5.24)$$

with

$$A := \frac{40c_2c_4c_5^{1/(q+4m)} E^{1/(q+4m)}}{\kappa}. \quad (5.25)$$

For the last inequality above, we have used an elementary estimate $5^\kappa/(5^\kappa - 1) \leq 2/\kappa$ which holds for each $\kappa \in (0, 1)$ ⁸.

Now, note that since $y - x \in H_1$ the initial cylinder F_1 is such that $F_1 \cap B^n(y, r_2)$ provides a (θ_1, H_1) -trapping box for Σ in $B^n(y, r_2)$. Thus, replacing the roles of x and y from the second step on, we may run a similar iteration and obtain a plane H_y^* such that

$$\angle(H_y^*, H_1) \leq Ar_1^\kappa, \quad (5.26)$$

together with a sequence of planes $P_N \rightarrow H_y^*$ (with $P_1 = H_1$) and appropriate trapping boxes determined by those planes. By the triangle inequality, (5.24) for $N = 1$ and (5.26) yield

$$\angle(H_x^*, H_y^*) \leq 2Ar_1^\kappa. \quad (5.27)$$

Once the uniqueness of H_x^* is established, we identify H_x^* with $T_x\Sigma$. The estimate (5.27) combined with (5.25) will yield the desired (5.14) (note that $r_1 \approx |x - y|$ up to a constant factor which is less than 2).

⁸Indeed, $f(\kappa) = \kappa a^\kappa \leq 2(a^\kappa - 1) = g(\kappa)$ for all $\kappa \in (0, 1)$ and $a > e$, as $f(0) = g(0)$ and $f' < g'$ on $(0, 1)$.

Step 3. Trapping boxes around H_x^* . It is now easy to check that tilting the cylinders F_N and enlarging them slightly, we can obtain new trapping boxes K_N for Σ in $B^n(x, 2r_N)$.

Fix $w \in F_N$. For sake of brevity, let Q_* and Q_N denote the orthogonal projections of \mathbb{R}^n onto $(H_x^*)^\perp$ and H_N^\perp . We have

$$\begin{aligned} |Q_*(w-x)| &= |Q_N(w-x) + (Q_*(w-x) - Q_N(w-x))| \\ &\leq 2c_4\varphi_N r_N + \angle(H_x^*, H_N)|w-x| \\ &\leq 2Ar_N^\kappa \cdot 2r_N, \end{aligned} \quad (5.28)$$

as $c_4\varphi_N \leq Ar_N^\kappa$. Hence, by (5.21),

$$|Q_*(w-x)| < 9Ad_N^\kappa \cdot d_N.$$

Therefore, if β_N is defined by (5.16) with

$$c_6 := 10AE^{-1/(q+4m)} = 400\kappa^{-1}c_2c_4c_5^{1/(q+4m)}, \quad (5.29)$$

then we have $|Q_*(w-x)| < \beta_N d_N$ for each $w \in F_N$.

Thus the cylinder

$$K_N := \{w \in B^n(x, 2d_N) : \text{dist}(w, x + H_x^*) \leq \beta_N \cdot 2d_N\}, \quad N = 1, 2, \dots \quad (5.30)$$

contains $F_N \cap B^n(x, 2d_N)$. It follows that $\Sigma \cap B^n(x, 2d_N) \subset K_N$, as F_N was a trapping box for Σ in a larger ball $B^n(x, 2r_N)$. It is easy to see that the linking condition of Definition 5.1 is also satisfied (we just take a smaller set of spheres that are slightly tilted) so that K_N indeed is a (β_N, H_x^*) -trapping box for Σ in $B^n(x, 2d_N)$.

Let us now specify a_2 . We choose this constant so that

$$c_6a_2^\kappa < \frac{1}{20} \quad \text{and} \quad 0 < a_2 < a_1, \quad (5.31)$$

where a_1 is the constant of Theorem 1.2. Then, by (5.13),

$$\begin{aligned} \beta_1 &= c_6E^{1/(q+4m)}|x-y|^\kappa \\ &< c_6E^{1/(q+4m)}R_2^\kappa \\ &= c_6E^{1/(q+4m)}a_2^\kappa E^{-1/(q+4m)} = c_6a_2^\kappa < \frac{1}{20}. \end{aligned} \quad (5.32)$$

For these choices of c_6 and a_2 all applications of Lemma 5.3 were justified. Now, returning to (5.27), we obtain

$$\begin{aligned} \angle(H_x^*, H_y^*) &\leq 2Ar_1^\kappa \\ &\stackrel{(5.29)}{=} \frac{c_6}{5}E^{1/(q+4m)}r_1^\kappa \\ &\stackrel{(5.17)}{<} \frac{c_6}{5}E^{1/(q+4m)}\left(\frac{3}{2}|x-y|\right)^\kappa \\ &< c_6E^{1/(q+4m)}|x-y|^\kappa. \end{aligned} \quad (5.33)$$

In particular, as $|x-y| < R_2$, we also have

$$\angle(H_x^*, H_y^*) < c_6E^{1/(q+4m)}R_2^\kappa. \quad (5.34)$$

To finish the whole proof, it remains to demonstrate that H_x^* is indeed unique and that $\Sigma \cap \text{int } B^n(x, R_2) = U$ is an open m -dimensional disk such that the projection $\pi_{H_x^*}|_U$ is injective.

Step 4. Uniqueness of H_x^* . Since formally Lemma 5.3 alone does not guarantee that the choice of each new plane H_N is unique, we must now show that $H_x^* = \lim H_N$ is unique.

Suppose that this were not the case, and that choosing $L_N \neq H_N$ in some steps of the iteration we could obtain a different limiting plane L , with $\angle(L, H_x^*) > 0$.

Select $w \in H_x^*$ with $|w| = 1$ such that $|w - \pi_L(w)| > \vartheta > 0$. Set $V := (H_x^*)^\perp$ and without loss of generality suppose that $x = 0$. The spheres

$$M_N := \mathbb{S}^{n-m-1}(d_N w, 3\beta_N d_N; V)$$

are contained in $\text{int } B^n(0, 2d_N)$, away from Σ since $\beta_N \leq \beta_1 < 1/20$, and by Lemma 3.2, are nontrivially linked with Σ since K_N is a (β_N, H_x^*) -trapping box for Σ in $B^n(0, 2d_N)$. Since L has been obtained by an analogous iteration process, the cylinders

$$\tilde{K}_N := \{w \in B^n(0, 2d_N) : \text{dist}(w, L) \leq \beta_N \cdot 2d_N\}$$

should also provide (β_N, L) -trapping boxes for Σ in $B^n(0, 2d_N)$. However, taking N so large that $6\beta_N < \vartheta$, we obtain $\text{dist}(d_N w, L) = d_N |w - \pi_L(w)| > d_N \vartheta > 6\beta_N d_N$. Thus, the sphere M_N is contained in the interior of $B^n(x, 2d_N) \setminus \tilde{K}_N$ and satisfies the assumptions of Lemma 3.4 with $\varepsilon = 2\beta_N d_N$ and therefore is *not* linked with Σ , a contradiction which proves that H_x^* has to be unique.

Moreover, since K_N is a (β_N, H_x^*) -trapping box for Σ in $B^n(x, 2d_N)$ and $\beta_N \approx d_N^\kappa$ one easily concludes that for $y \in \Sigma$ we have

$$\text{dist}(y, x + H_x^*) = O(|x - y|^{1+\kappa}) \quad \text{as } y \rightarrow x,$$

which justifies the definition $T_x \Sigma := H_x^*$.

Step 5. Injectivity of the projection. Again, we argue by contradiction. Suppose that there exist $y \neq y_1 \in U \equiv \Sigma \cap \text{int } B^n(x, R_2)$ such that $\pi_{T_x \Sigma}(y) = \pi_{T_x \Sigma}(y_1)$. Without loss of generality suppose that

$$|x - y_1| \leq |x - y| = d < R_2;$$

and let d_N, β_N be defined by (5.16). Set $v = y_1 - y$ and let $Q_{T_x \Sigma}, Q_{T_y \Sigma}$ denote the projections onto $(T_x^\Sigma)^\perp = (H_x^*)^\perp, (T_y \Sigma)^\perp = (H_y^*)^\perp$, respectively. As $v \perp T_x \Sigma$, we have $Q_{T_x \Sigma}(v) = v$ and

$$\begin{aligned} |Q_{T_y \Sigma}(v)| &= |Q_{T_x \Sigma}(v) + (Q_{T_y \Sigma}(v) - Q_{T_x \Sigma}(v))| \\ &\geq |v|(1 - \|Q_{T_x \Sigma} - Q_{T_y \Sigma}\|) \\ &= |v|(1 - \angle(H_x^*, H_y^*)) \\ &\geq |v|(1 - c_6 E^{1/(q+4m)} R_2^\kappa) \quad \text{by (5.34)} \\ &= |v|(1 - c_6 a_2^\kappa) > \frac{19}{20} |v| \end{aligned}$$

by (5.31). However, fixing N so that $d_{N+1} < |v| = |y - y_1| \leq d_N$, we could use the trapping boxes constructed along with H_y^* , i.e. the cylinders

$$\{w \in B^n(y, 2d_N) : \text{dist}(w, y + H_y^*) \leq \beta_N \cdot 2d_N\}$$

which contain $\Sigma \cap B^n(y, 2d_N)$, to estimate by virtue of (5.32)

$$|Q_{T_y \Sigma}(v)| \leq 2\beta_N d_N = 10\beta_N d_{N+1} \leq 10\beta_1 d_{N+1} < \frac{1}{2} |v|,$$

a contradiction.

Since for each $d_1 < R_2$ the cylinder

$$K_1 = \{w \in B^n(x, 2d_1) : \text{dist}(w, x + H_x^*) \leq 2\beta_1 d_1\} \quad (5.35)$$

is a (β_1, H_x^*) -trapping box for Σ in $B^n(x, 2d_1)$, and $\beta_1 < 1/20$, we know by now – as d_1 can be taken very close to R_2 – that the image of $\pi_{T_x \Sigma}$ restricted to, say, $\Sigma \cap B^n(x, 3R_2/2)$, certainly contains the disk with center at $\pi_{T_x \Sigma}(x)$ and radius R_2 . It follows that $U = \text{int} B^n(x, R_2) \cap \Sigma$ is a topological disk, since $\pi_{T_x \Sigma}$ was also shown to be injective. \square

Proof of Theorem 1.3. As $\beta_1 = c_6 E^{1/(q+4m)} d_1^\kappa$, it can be checked that Theorem 1.3 stated in the introduction follows from (5.35) and the definition of a trapping box. One can use the plane $H_x^* \in G(n, m)$ to estimate the infimum in the definition (1.5) of β -numbers. \square

5.4 Local graph representations of Σ

We shall now use Theorem 5.6 to construct the graph representations of an admissible surface Σ with $\mathcal{E}_q(\Sigma) < \infty$ for some $q > 2m$. In Section 6, this will be used to show that Σ is in fact a manifold of class $C^{1,\mu}$ for $\mu = 1 - 2m/q > \kappa$.

Corollary 5.7. *Suppose that $\Sigma \in \mathcal{A}(\delta)$ for some $\delta \in (0, \delta(m)]$, $\mathcal{E}_q(\Sigma) \leq E$, $q > 2m$. Let $a_2 > 0$ and $R_2 = a_2 E^{-1/(q-2m)}$ denote the constants introduced in Theorem 5.6. Set $R_3 = \frac{1}{2}R_2$. Then, for each $x \in \Sigma$, the following is true.*

There exists a function

$$f : T_x \Sigma =: P \simeq \mathbb{R}^m \rightarrow P^\perp \simeq \mathbb{R}^{n-m}$$

of class $C^{1,\kappa}$, $\kappa = \frac{q-2m}{q+4m}$, such that $f(0) = 0$ and $\nabla f(0) = 0$, and

$$\Sigma \cap B^n(x, R_3) = x + \left(\text{graph } f \cap B^n(0, R_3) \right),$$

where $\text{graph } f \subset P \times P^\perp = \mathbb{R}^n$ denotes the graph of f , and

$$|\nabla f(z) - \nabla f(w)| \leq c_7 E^{1/(q+4m)} |z - w|^\kappa \leq c_7 E^{1/(q+4m)} (2R_3)^\kappa, \quad z, w \in P \cap B^n(x, R_3), \quad (5.36)$$

for some constant c_7 depending only on n, m, q .

Proof. Without loss of generality suppose that $x = 0 \in \mathbb{R}^n$ and $T_x \Sigma = P = \text{span}(e_1, \dots, e_m)$, where e_j , $j = 1, \dots, n$, form the standard orthonormal basis of \mathbb{R}^n . By Theorem 5.6 we know that

$$\pi_P|_U : \Sigma \cap B^n(x, R_2) \rightarrow \pi(U) \subset P, \quad U := \Sigma \cap \text{int} B^n(x, R_2),$$

is invertible. By (5.15) and (5.16) for $N = 1$, the image of this map contains an m -dimensional disk of radius $R'_2 = R_2^2 - (R_2/10)^2 > \frac{9}{10}R_2$.

Step 1. We now let

$$f : = Q_P \circ \left(\pi_P|_U \right)^{-1} \Big|_D : D \rightarrow P^\perp, \quad D = \text{int } D^m(0, R'_2) \subset P,$$

so that

$$D \ni z \longmapsto F(z) := (z, f(z)) \in P \times P^\perp = \mathbb{R}^n$$

is a natural parametrization of Σ . Note that $F(D)$ contains $\Sigma \cap B^n(x, R_3)$ and that $f(0) = 0$. Both f and F are continuous.

Step 2. To prove that $\nabla f(0)$ exists and equals 0, use now the definition of f to see that (5.13), (5.15) and (5.16) of Theorem 5.6 yield

$$|f(z)| \stackrel{(5.15)}{\leq} 2\beta_N d_N \stackrel{(5.16)}{\leq} C(n, m, q, E) d_N^{1+\kappa} \quad \text{for all } N \in \mathbb{N},$$

whenever $F(z) = (z, f(z)) \in B^n(0, 2d_N)$. (Recall that $d_N = d_1 \cdot 5^{1-N}$; we are free to use any $d_1 < R_2$ here.) Set $\rho_N := d_N(1 - \beta_N^2)^{1/2}$; by (5.16), $\frac{19}{20}d_N < \rho_N \leq d_N$. Thus, we also have $|f(z)| \leq \text{const} \cdot \rho_N^{1+\kappa}$ whenever $z \in D^m(0, 2\rho_N) \subset P$. As $\rho_N \approx d_N = d_1 5^{1-N}$ for $N = 1, 2, \dots$, this gives $|f(z)| = O(|z|^{1+\kappa})$ near 0 and consequently $\nabla f(0) = 0$.

We shall now show that F (and hence f) is differentiable at each $z \in D$. Fix $z \in D$ and $h \in P$ with $|h|$ small. Set

$$L := \left(\pi_P \Big|_{T_{F(z)}\Sigma} \right)^{-1} : P \rightarrow T_{F(z)}\Sigma \hookrightarrow \mathbb{R}^n.$$

We have $F(z+h) - F(z) = L(h) + e$, where the error $e = F(z+h) - F(z) - L(h)$ satisfies, by definition of L and F , $\pi_P(e) = 0$. Thus, $e = Q_P(e)$, so that

$$|e| \leq \left| (Q_P - Q_{T_{F(z)}\Sigma})(e) \right| + \left| Q_{T_{F(z)}\Sigma}(e) \right| \leq \frac{1}{20}|e| + \left| Q_{T_{F(z)}\Sigma}(e) \right| \quad \text{by (5.14) and (5.16) for } N = 1.$$

Absorbing the first term and using now Theorem 5.6 at $x = F(z)$, we obtain

$$|e| \leq \frac{20}{19} \left| Q_{T_{F(z)}\Sigma}(e) \right| = \frac{20}{19} \text{dist}(F(z+h), F(z) + T_{F(z)}\Sigma) = O(|F(z+h) - F(z)|^{1+\kappa}). \quad (5.37)$$

To finish the estimates, note that

$$|L(h) - h| = |\pi_{T_{F(z)}\Sigma}(L(h)) - \pi_P(L(h))| \stackrel{(5.14)}{\leq} \frac{1}{20}|L(h)|;$$

therefore, $\frac{19}{20}|L(h)| \leq |h| \leq \frac{21}{20}|L(h)|$. Using this and (5.37), we now write

$$|F(z+h) - F(z)| \leq |L(h)| + |e| \leq \frac{20}{19} \left(|h| + \left| Q_{T_{F(z)}\Sigma}(e) \right| \right) \leq \frac{20}{19} (|h| + \text{const} \cdot |F(z+h) - F(z)|^{1+\kappa}).$$

Now, for all $|h|$ sufficiently small we have $\frac{20}{19} \text{const} \cdot |F(z+h) - F(z)|^{1+\kappa} < \frac{1}{2}|F(z+h) - F(z)|$, as F is continuous at z . Thus, the second term can be absorbed, yielding $|F(z+h) - F(z)| = O(|h|)$ as $h \rightarrow 0$. Plugging this into the right hand side of (5.37), we obtain the desired error estimate $|e| = O(|h|^{1+\kappa}) = o(|h|)$ as $h \rightarrow 0$. Therefore, F is differentiable at z with $DF(z) = L$.

The uniform Hölder bound for ∇f results now from one more application of the oscillation estimate (5.14) for tangent planes:

Step 3. With

$$|\partial_i f(w) - \partial_i f(z)| = \left| \begin{bmatrix} e_i \\ \partial_i f(w) \end{bmatrix} - \begin{bmatrix} e_i \\ \partial_i f(z) \end{bmatrix} \right| = \left| \pi_{T_{F(w)}\Sigma} \left(\begin{bmatrix} e_i \\ \partial_i f(w) \end{bmatrix} \right) - \pi_{T_{F(z)}\Sigma} \left(\begin{bmatrix} e_i \\ \partial_i f(z) \end{bmatrix} \right) \right|$$

we can estimate

$$\begin{aligned}
|\partial_i f(w) - \partial_i f(z)| &\leq \left| \pi_{T_{F(w)}\Sigma} \left(\begin{bmatrix} e_i \\ \partial_i f(w) \end{bmatrix} \right) - \pi_{T_{F(z)}\Sigma} \left(\begin{bmatrix} e_i \\ \partial_i f(w) \end{bmatrix} \right) \right| \\
&\quad + \left| \pi_{T_{F(z)}\Sigma} \left(\begin{bmatrix} e_i \\ \partial_i f(w) \end{bmatrix} \right) - \pi_{T_{F(z)}\Sigma} \left(\begin{bmatrix} e_i \\ \partial_i f(z) \end{bmatrix} \right) \right| \\
&\leq \angle(T_{F(w)}\Sigma, T_{F(z)}\Sigma)(1 + |\nabla f(w)|^2)^{1/2} + \left| (\pi_{T_{F(z)}\Sigma} - \pi_{T_0\Sigma}) \left(\begin{bmatrix} 0 \\ \partial_i f(w) - \partial_i f(z) \end{bmatrix} \right) \right| \\
&\stackrel{(5.14)}{<} c_6 E^{1/(q+4m)} |w - z|^\kappa + \angle(T_{F(z)}\Sigma, T_0\Sigma) |\partial_i f(w) - \partial_i f(z)|.
\end{aligned}$$

Since $\angle(T_{F(z)}\Sigma, T_0\Sigma) < 1/2$ by (5.14) and our choice of constants, we can absorb the right term on the left-hand side to conclude.

Now, using a standard cutoff technique, we leave f unchanged on $D^m(0, 2R_2/3)$, and extend it to the whole plane P , so that the extension vanishes off $D^m(0, 3R_2/4)$. The corollary follows. \square

6 Slicing and bootstrap to optimal Hölder exponent

In this section we assume that Σ is a flat m -dimensional graph of class $C^{1,\kappa}$ having finite tangent-point energy $\mathcal{E}_q(\Sigma)$. The goal is to show how to bootstrap the Hölder exponent κ to $\mu = 1 - 2m/q$.

Relying on Corollary 5.7, without loss of generality we can assume that

$$\Sigma \cap B^n(0, 5R) = \text{Graph } f \cap B^n(0, 5R)$$

for a fixed number $R > 0$, where

$$f: P \cong \mathbb{R}^m \rightarrow P^\perp \cong \mathbb{R}^{n-m}$$

is of class $C^{1,\kappa}$ and satisfies $\nabla f(0) = 0$, $f(0) = 0$,

$$|\nabla f| \leq \varepsilon_0 := \frac{\varepsilon_1}{800mc_2} = 2^{-5} 10^{-3} m^{-2} (10^m + 1)^{-2} \quad \text{on } P. \quad (6.1)$$

To achieve (6.1), we use (5.36) of Corollary 5.7 and shrink R_3 by a constant factor if necessary. The number ε_0 is chosen so that $\varepsilon_0 < \varepsilon_1/(400mc_2)$ for the constants ε_1 and c_2 used in Lemma 2.3 and other auxiliary estimates in Section 2.2. We let $F: P \rightarrow \mathbb{R}^n$ be the natural parametrization of $\Sigma \cap B(0, 5R)$, given by $F(x) = (x, f(x))$ for $x \in P$; outside $B^n(0, 5R)$ the image of F does not have to coincide with Σ . The choice of ε_0 guarantees that, due to Lemma 2.3 (ii),

$$\angle(T_{F(x)}\Sigma, T_{F(0)}\Sigma) \leq c_2 \varepsilon_0 < \frac{\varepsilon_1}{400m} \quad (6.2)$$

whenever $x \in B^n(0, 5R) \cap P$. Thus,

$$\angle(T_{F(x_1)}\Sigma, T_{F(x_2)}\Sigma) < \frac{\varepsilon_1}{200m} < \frac{1}{m4^{m+1}} \quad \text{for all } x_1, x_2 \in B^n(0, 5R) \cap P. \quad (6.3)$$

As in our paper [34, Section 6], we introduce the maximal functions controlling the oscillation of ∇f at various places and scales,

$$\Phi^*(\rho, A) = \sup_{B_\rho \subset A} \left(\text{osc}_{B_\rho} \nabla f \right) \quad (6.4)$$

where the supremum is taken over all possible closed m -dimensional balls B_ρ of radius ρ that are contained in a subset $A \subset B^n(0, 5R) \cap P$, with $\rho \leq 5R$. Since $f \in C^{1,\kappa}$, we have a priori

$$\Phi^*(\rho, A) \leq C\rho^\kappa \quad (6.5)$$

for some constant C which does not depend on ρ, A .

To show that $f \in C^{1,\mu}$ for $\mu = 1 - 2m/q$, we check that locally, on each scale, the oscillation of ∇f is controlled by a main term which involves the local energy and resembles the right hand side of (1.6), up to a small error, which itself is controlled by the oscillation of ∇f on a much smaller scale.

Lemma 6.1. *Let $f, F, \Sigma, R > 0$ and P be as above. If $z_1, z_2 \in B^n(0, 2R) \cap P$ with $|z_1 - z_2| = t > 0$, then for any $N > 2$ we have*

$$|\nabla f(z_1) - \nabla f(z_2)| \leq 2\Phi^*(t/N, B) + C(N, m, q) E_B^{1/q} t^\mu \quad (6.6)$$

where $B := B^m(\frac{z_1 + z_2}{2}, t)$ is an m -dimensional disc in P , $\mu := 1 - 2m/q$, and

$$E_B = \iint_{F(B) \times F(B)} R_{\text{tp}}^{-q} d\mathcal{H}^m \otimes d\mathcal{H}^m \quad (6.7)$$

is the local energy of Σ over B .

Remark. Once this lemma is proved, one can fix an m -dimensional disk $B^m(b, s) \subset B^n(0, R) \cap P$ and use (6.6) to obtain for $t \leq s$

$$\Phi^*(t, B^m(b, s)) \leq 2\Phi^*(2t/N, B^m(b, s+2t)) + C(N, m, q) M_q(b, s+2t) t^\mu \quad (6.8)$$

where

$$M_q(b, r) := \left(\iint_{F(B(b,r)) \times F(B(b,r))} R_{\text{tp}}^{-q} d\mathcal{H}^m \otimes d\mathcal{H}^m \right)^{1/q}.$$

Fixing $N > 2$ such that $2^\kappa/N^\kappa < \frac{1}{2}$ we obtain $2^j \cdot (2/N)^{j\kappa} \rightarrow 0$ as $j \rightarrow \infty$. Using this, one can iterate (6.8) and show that

$$\text{osc}_{B^m(b,s)} \nabla f \leq C'(m, q) M_q(b, 5s) s^\mu.$$

Combining this estimate with Corollary 5.7, we obtain Theorem 1.4 stated in the introduction. Note that in fact the result holds for all surfaces $\Sigma \in \mathcal{A}(\delta)$ for $\delta \in (0, \delta(m)]$, where $\delta(m)$ is the constant of Lemma 4.3.

The remaining part of this section is devoted to the

Proof of Lemma 6.1. Fix z_1, z_2 and the disk B as in the statement of the lemma; we have $\mathcal{H}^m(B) = \omega(m)t^m$. Pick $N > 2$ and let E_B be the local energy of Σ over B , defined by (6.7). Assume that $\nabla f \not\equiv \text{const}$ on B , for otherwise there is nothing to prove.

Step 1. Take

$$K_0 := (E_B \cdot N^{2m} \omega(m)^{-2})^{1/q} > 0 \quad (6.9)$$

and set

$$Y_1 := \{x_1 \in B : \mathcal{H}^1(Y_2(x_1)) \geq N^{-m} \mathcal{H}^m(B)\}, \quad (6.10)$$

$$Y_2(x_1) := \left\{x_2 \in B : \frac{1}{R_{\text{tp}}(F(x_1), F(x_2))} > K_0 t^{-2m/q}\right\}. \quad (6.11)$$

We now estimate the local energy to obtain a bound for $\mathcal{H}^m(Y_1)$, shrinking the domain of integration, as follows:

$$\begin{aligned}
E_B &= \iint_{F(B) \times F(B)} R_{\text{tp}}^{-q} d\mathcal{H}^m \otimes d\mathcal{H}^m \\
&\geq \iint_{B \times B} \left(\frac{1}{R_{\text{tp}}(F(x_1), F(x_2))} \right)^q dx_1 dx_2 \\
&\geq \int_{Y_1} \left(\int_{Y_2(x_1)} \left(\frac{1}{R_{\text{tp}}(F(x_1), F(x_2))} \right)^q dx_2 \right) dx_1 \\
&\stackrel{(6.10), (6.11)}{>} \mathcal{H}^m(Y_1) N^{-m} \mathcal{H}^m(B) K_0^q t^{-2m} = E_B \mathcal{H}^m(Y_1) N^m (\mathcal{H}^m(B))^{-1}.
\end{aligned}$$

The last equality follows from (6.9). Thus, we obtain

$$\mathcal{H}^m(Y_1) < \frac{1}{N^m} \mathcal{H}^m(B),$$

and since the radius of B equals t , we obtain

$$B^m(a_i, t/N) \setminus Y_1 \neq \emptyset \quad \text{for } i = 1, 2. \quad (6.12)$$

Now, select two points $u_i \in B^m(a_i, t/N) \setminus Y_1$ ($i = 1, 2$). By the triangle inequality,

$$\begin{aligned}
|\nabla f(z_1) - \nabla f(z_2)| &\leq |\nabla f(z_1) - \nabla f(u_1)| + |\nabla f(u_2) - \nabla f(z_2)| + |\nabla f(u_1) - \nabla f(u_2)| \\
&\leq 2\Phi^*(t/N, B) + |\nabla f(u_1) - \nabla f(u_2)|.
\end{aligned}$$

Thus, it remains to show that the last term, $|\nabla f(u_1) - \nabla f(u_2)|$, does not exceed a constant multiple of $E_B^{1/q} t^\mu$. To achieve this goal, we assume that $\nabla f(u_1) \neq \nabla f(u_2)$ and work with the portion of the surface parametrized by the points in

$$G := B \setminus (Y_2(u_1) \cup Y_2(u_2)). \quad (6.13)$$

By (6.10), G satisfies

$$\mathcal{H}^m(G) > (1 - 2N^{-m}) \mathcal{H}^m(B) =: C_1(q, m) t^m. \quad (6.14)$$

To conclude the whole proof, we shall derive an upper estimate for the measure of G ,

$$\mathcal{H}^m(G) \leq C_2(q, m) K_0 \frac{t^{m+\mu}}{\alpha}, \quad (6.15)$$

where $\alpha := \angle(H_1, H_2) \neq 0$ and $H_i := T_{F(u_i)} \Sigma$ denotes the tangent plane to Σ at $F(u_i) \in \Sigma$ for $i = 1, 2$. Combining (6.15) and (6.14), we will then obtain

$$\alpha < (C_1)^{-1} C_2 K_0 t^\mu =: C_3 E_B^{1/q} t^\mu.$$

(By a reasoning analogous to the proof of Corollary 5.7, this also yields an estimate for the oscillation of ∇f .)

Step 2. Proof of (6.15). By (6.3), we have $\alpha = \angle(H_1, H_2) < m^{-1} 4^{-m-1}$. By Lemma 2.8 applied to $\varepsilon = m^{-1} 4^{-m-1}$, we obtain

$$\mathcal{H}^m(G) \leq \mathcal{H}^m(F(G)) < 2\mathcal{H}^m(\pi_{H_1}(F(G))),$$

so that (6.15) would follow from

$$\mathcal{H}^m(\pi_{H_1}(F(G))) \leq C_4 K_0 \frac{t^{m+\mu}}{\alpha}. \quad (6.16)$$

Now, for $\zeta \in G$ and $i = 1, 2$ we have by (6.11)

$$\frac{1}{R_{\text{tp}}(F(u_i), F(\zeta))} = \frac{2|Q_{H_i}(F(\zeta) - F(u_i))|}{|F(\zeta) - F(u_i)|^2} \leq K_0 t^{-1+\mu}.$$

Let $P_i = F(u_i) + H_i$ be the affine tangent plane to Σ at $F(u_i)$. Since F is Lipschitz with constant $(1 + \varepsilon_0)$ and $|z - u_i| \leq 2t$,

$$\begin{aligned} \text{dist}(F(\zeta), P_i) &= \text{dist}(F(\zeta) - F(u_i), H_i) \\ &= |Q_{H_i}(F(\zeta) - F(u_i))| \leq 8K_0 t^{1+\mu} =: h_0 \end{aligned} \quad (6.17)$$

for $\zeta \in G, i = 1, 2$. Select the points $p_i \in P_i, i = 1, 2$, so that $|p_1 - p_2| = \text{dist}(P_1, P_2)$. The vector $p_2 - p_1$ is then orthogonal to H_1 and to H_2 , and since G is nonempty by (6.14), we have $|p_1 - p_2| \leq 2h_0$ by (6.17).

Set $p = (p_1 + p_2)/2$, pick a parameter $\zeta \in G$ and consider $y = F(\zeta) - p$. We have

$$y = (F(\zeta) - F(u_1)) + (F(u_1) - p_1) + (p_1 - p),$$

so that $\pi_{H_1}(y) = \pi_{H_1}(F(\zeta) - F(u_1)) + (F(u_1) - p_1)$, and

$$\begin{aligned} |y - \pi_{H_1}(y)| &= |(p_1 - p) + F(\zeta) - F(u_1) - \pi_{H_1}(F(\zeta) - F(u_1))| \\ &= |(p_1 - p) + Q_{H_1}(F(\zeta) - F(u_1))|. \end{aligned}$$

Therefore, since $|p - p_1| \leq h_0$ and by (6.17), $|y - \pi_{H_1}(y)| \leq h_0 + h_0 = 2h_0$. In the same way, we obtain $|y - \pi_{H_2}(y)| \leq 2h_0$. Thus,

$$\frac{y}{2h_0} = \frac{F(\zeta) - p}{2h_0} \in S(H_1, H_2),$$

where $S(H_1, H_2) = \{x \in \mathbb{R}^n : \text{dist}(x, H_i) \leq 1 \text{ for } i = 1, 2\}$ is the intersection of two slabs considered in Section 2.2. Applying Lemma 2.6 which is possible due to the estimate (6.3) for $\angle(H_1, H_2)$, we conclude that there exists an $(m - 1)$ -dimensional subspace $W \subset H_1$ such that

$$\pi_{H_1}(F(G) - p) \subset \{x \in H_1 : \text{dist}(x, W) \leq 2h_0 \cdot 5c_2/\alpha\}. \quad (6.18)$$

On the other hand, since F is Lipschitz, we certainly have $F(G) \subset B^n(F(\frac{a_1+a_2}{2}), 2t)$ and therefore

$$\pi_{H_1}(F(G) - p) \subset B^n(a, 2t), \quad a := \pi_{H_1}(F(\frac{a_1+a_2}{2}) - p). \quad (6.19)$$

Combining (6.18) and (6.19), we invoke Lemma 2.7 to $H := H_1$, $S' := \pi_{H_1}(F(G) - p)$, and $d := 2h_0 5c_2/\alpha$, to obtain

$$\mathcal{H}^m(\pi_{H_1}(F(G))) \leq 4^{m-1} t^{m-1} \cdot 20h_0 c_2/\alpha =: C_2(m) K_0 \frac{t^{m+\mu}}{\alpha},$$

which is (6.16), implying (6.15) and thus completing the proof.

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